A POLYNOMIAL CANONICAL FORM FOR CERTAIN ANALYTIC FUNCTIONS OF TWO VARIABLES AT A CRITICAL POINT

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THEOREM. Let $F(z, w)$ be analytic for small $|z|$ and $|w|$ and $F(0, 0) = 0$. Then (Weierstrass Preparation Theorem)

$$F(z, w) = z^k[w^m + a_1(z)w^{m-1} + \cdots + a_m(z)]\Phi(z, w)$$

where $\Phi(0, 0) \neq 0$ and $a_j(0) = 0$. Let the discriminant of the polynomial in $w$, in the bracket above, not vanish identically (so that there are no quadratic factors of $F$ which are polynomials in $w$). Then there exists $\psi(\zeta, \omega)$ a polynomial in $(\zeta, \omega)$ of degree $m$ in $\omega$ and analytic functions $\gamma(z, w)$ and $\delta(z, w)$ such that $\gamma(0, 0) = \delta(0, 0) = 0$ and similarly for $\delta$ such that if

$$\zeta = z + \gamma(z, w), \quad \omega = w + \delta(z, w)$$

then

$$\psi(\zeta, \omega) = F(z, w).$$

(Note that $\psi$ is a polynomial in both variables.) An outline of the proof follows.

By [1] it is known that $F$ can be transformed by use of (2) to the form of (1) with $\Phi = 1$. Hence the case

$$F(z, w) = f_0(z)w^m + f_1(z)w^{m-1} + \cdots + f_m(z)$$

where $f_0 = z^k$ and $z^{k+1}|f_j(z)$ $j \geq 1$, can be considered.

Because of the hypothesis on $F$ it can be shown that the resultant of $F_z = \partial F/\partial z$ and $F_w$ does not vanish identically. Thus

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for small $|z| > 0$. Let the lowest nonvanishing power of $z$ in $D(z)$ be $z^n$. Let $P_0(z) = z^k$ and for $j \geq 1$ let $P_j(z)$ be polynomials of degree $2\mu + 2$ such that $P_j - f_j$ has least power of $z$ of degree at least $2\mu + 3$. Let the polynomial

$$\psi(z, \omega) = z^k \omega^m + P_1(z) \omega^{m-1} + \cdots + P_m(z).$$

Consider now the equation

$$\psi(z + g, w + h) = (z + g) \psi(z, w) + g \psi_1(z, w) + h \psi_2(z, w) + R(z, w, g, h)$$

where each term in the polynomial $R$ is of degree at least two in $(g, h)$. Hence (8) can be written as

$$(g_0 + w g_1 + \cdots + w^{m-2} g_{m-2}) \psi_1(z, w) + (h_0 + \cdots + w^{m-1} h_{m-1}) \psi_2(z, w) = (f_1 - P_1) w^{m-1} + \cdots + (f_m - P_m)$$

$$- R(z, w, g_0 + \cdots + g_{m-2} w^{m-2}, h_0 + \cdots ).$$

Certainly the equation (7) will be satisfied if the coefficients of $w^i$ on the left are set equal to those of $w^i$ on the right except that $- R$ is kept with $f_m - P_m$ so that the $2m - 1$ equations obtained from (7) are

$$(8) \quad P_0'(z) g_{m-2} + m P_0(z) h_{m-1} = 0, \cdots ,$$

$$P_m' g_0 + P_{m-1} h_0 = f_m - P_m - R.$$

Because of (5) and the coincidence of the early terms of $P_j$ and $f_j$, the equations (8) can be inverted to give

$$(9) \quad g_i = z^{- \nu} \sum_{j=1}^{m} \alpha_{ij}(z) (f_j - P_j) - z^{- \nu} \alpha_{im} R,$$

$$i = 0, \cdots , m - 2$$

$$h_i = z^{- \nu} \sum_{j=1}^{m} \beta_{ij}(z) (f_j - P_j) - z^{- \nu} \beta_{im} R,$$

$$i = 0, \cdots , m - 1$$
where \( \alpha_{ij} \) and \( \beta_{ij} \) are analytic in \( z \). Next let \( g_i = z^{r+1}u_i \) and \( h_i = z^{r+1}v_i \). If

\[
R(z, w, z^{r+1}u_0 + \cdots + z^{r+1}u_{m-2}w^{m-2}, z^{r+1}v_0 + \cdots + z^{r+1}v_{m-1}w^{m-1})
= z^{2r+2} \tilde{R}(z, w, u_0, \cdots, u_{m-2}, v_0, \cdots, v_{m-1})
\]

then \( \tilde{R} \) is a polynomial in all variables of degree at least two in \((u_i, v_i)\). Hence (9) and (10) become

\[
\begin{align*}
  u_i + z\alpha_{im}(z)\tilde{R} &= \sum_{i=1}^{m} \alpha_{ij}(z)z^{-2r-1}(f_i - P_i), & i = 0, \cdots, m - 2, \\
  v_i + z\beta_{im}\tilde{R} &= \sum_{i=1}^{m} \beta_{ij}(z)z^{-2r-1}(f_i - P_i), & i = 0, \cdots, m - 1.
\end{align*}
\]

Since \((f_i - P_i)z^{-2r-1}\) is analytic and vanishes at \( z=0 \), and since \( u_i = v_i = 0 \) is a solution of (11) for \( z=w=0 \), it follows from the implicit function theorem that for small \( |z| \) and \( |w| \), (11) has an analytic solution \( u_i(z, w), v_i(z, w) \).

The question of whether it was possible to extend the result of [1] to the form of a polynomial \( \psi \) in both variables (rather than in just one as in [1]) was asked of me by Felix Browder.

**Reference**