RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A POLYNOMIAL CANONICAL FORM FOR CERTAIN ANALYTIC FUNCTIONS OF TWO VARIABLES AT A CRITICAL POINT[1]

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THEOREM. *Let $F(z, w)$ be analytic for small $|z|$ and $|w|$ and $F(0, 0) = 0$. Then (Weierstrass Preparation Theorem)*

$$F(z, w) = z^k[w^m + a_1(z)w^{m-1} + \cdots + a_m(z)]\Phi(z, w) \tag{1}$$

where $\Phi(0, 0) \neq 0$ and $a_j(0) = 0$. Let the discriminant of the polynomial in w, in the bracket above, not vanish identically (so that there are no quadratic factors of F which are polynomials in w). Then there exists $\psi(\zeta, \omega)$ a polynomial in (ζ, ω) of degree m in ω and analytic functions $\gamma(z, w)$ and $\delta(z, w)$ such that $\gamma(0, 0) = \partial\gamma/\partial z(0, 0) = \partial\gamma/\partial w(0, 0) = 0$ and similarly for δ such that if

$$\zeta = z + \gamma(z, w), \qquad \omega = w + \delta(z, w) \tag{2}$$

then

$$\psi(\zeta, \omega) = F(z, w). \tag{3}$$

(Note that ψ is a polynomial in both variables.) An outline of the proof follows.

By [1] it is known that F can be transformed by use of (2) to the form of (1) with $\Phi = 1$. Hence the case

$$F(z, w) = f_0(z)w^m + f_1(z)w^{m-1} + \cdots + f_m(z) \tag{4}$$

where $f_0 = z^k$ and $z^{k+1} | f_j(z)$ $j \geqq 1$, can be considered.

Because of the hypothesis on F it can be shown that the resultant of $F_z = \partial F/\partial z$ and F_w does not vanish identically. Thus

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$$(5)\qquad D(z) = \begin{vmatrix} 0 & 0 & \cdots 0 & f_0' & 0 & \cdots 0 & & mf_0 \\ 0 & 0 & \cdots f_0' & f_1' & 0 & \cdots mf_0 & & (m-1)f_1 \\ & & & \vdots & & & & \\ f_{m-1}' & f_m' & \cdots 0 & 0 & & 2f_{m-2} & \cdots 0 & 0 \\ f_m' & 0 & \cdots 0 & 0 & & f_{m-1} & \cdots 0 & 0 \end{vmatrix} \neq 0$$

for small $|z| > 0$. Let the lowest nonvanishing power of z in $D(z)$ be z^μ. Let $P_0(z) = z^k$ and for $j \geqq 1$ let $P_j(z)$ be polynomials of degree $2\mu + 2$ such that $P_j - f_j$ has least power of z of degree at least $2\mu+3$. Let the polynomial

$$\psi(\zeta, \omega) = \zeta^k \omega^m + P_1(\zeta)\omega^{m-1} + \cdots + P_m(\zeta).$$

Consider now the equation

$$(6)\qquad \begin{aligned} \psi(z + g_0 + wg_1 + \cdots + w^{m-2}g_{m-2}, w + h_0 + \cdots + w^{m-1}h_{m-1}) \\ = P_0(z)w^m + f_1(z)w^{m-1} + \cdots + f_m(z). \end{aligned}$$

Clearly

$$\psi(z + g, w + h) = \psi(z, w) + g\psi_z(z, w) + h\psi_w(z, w) + R(z, w, g, h)$$

where each term in the polynomial R is of degree at least two in (g, h). Hence (8) can be written as

$$(7)\qquad \begin{aligned} (g_0 + wg_1 + \cdots + w^{m-2}g_{m-2})\psi_z(z, w) \\ + (h_0 + \cdots + w^{m-1}h_{m-1})\psi_w(z, w) \\ = (f_1 - P_1)w^{m-1} + \cdots + (f_m - P_m) \\ - R(z, w, g_0 + \cdots + g_{m-2}w^{m-2}, h_0 + \cdots). \end{aligned}$$

Certainly the equation (7) will be satisfied if the coefficients of w^j on the left are set equal to those of w^j on the right except that $-R$ is kept with $f_m - P_m$ so that the $2m-1$ equations obtained from (7) are

$$(8)\qquad \begin{aligned} P_0'(z)g_{m-2} + mP_0(z)h_{m-1} = 0, \cdots, \\ P_m' g_0 + P_{m-1}h_0 = f_m - P_m - R. \end{aligned}$$

Because of (5) and the coincidence of the early terms of P_j and f_j, the equations (8) can be inverted to give

$$(9)\qquad g_i = z^{-\mu} \sum_{j=1}^{m} \alpha_{ij}(z)(f_j - P_j) - z^{-\mu}\alpha_{im}R, \qquad i = 0, \cdots, m-2$$

$$(10)\qquad h_i = z^{-\mu} \sum_{j=1}^{m} \beta_{ij}(z)(f_j - P_j) - z^{-\mu}\beta_{im}R, \qquad i = 0, \cdots, m-1$$

where α_{ij} and β_{ij} are analytic in z. Next let $g_i = z^{\mu+1}u_i$ and $h_i = z^{\mu+1}v_i$. If

$$R(z, w, z^{\mu+1}u_0 + \cdots + z^{\mu+1}u_{m-2}w^{m-2}, z^{\mu+1}v_0 + \cdots + z^{\mu+1}v_{m-1}w^{m-1}) = z^{2\mu+2}\tilde{R}(z, w, u_0, \cdots, u_{m-2}, v_0, \cdots, v_{m-1})$$

then $\tilde{R}$ is a polynomial in all variables of degree at least two in (u_i, v_j). Hence (9) and (10) become

$$\begin{aligned} u_i + z\alpha_{im}(z)\tilde{R} &= \sum_{i=1}^{m} \alpha_{ij}(z)z^{-2\mu-1}(f_i - P_i), \quad i = 0, \cdots, m-2, \\ v_i + z\beta_{im}\tilde{R} &= \sum_{i=1}^{m} \beta_{ij}(z)z^{-2\mu-1}(f_i - P_i), \quad i = 0, \cdots, m-1. \end{aligned} \tag{11}$$

Since $(f_i - P_i)z^{-2\mu-1}$ is analytic and vanishes at $z=0$, and since $u_i = v_i = 0$ is a solution of (11) for $z = w = 0$, it follows from the implicit function theorem that for small $|z|$ and $|w|$, (11) has an analytic solution $u_i(z, w)$, $v_i(z, w)$.

The question of whether it was possible to extend the result of [1] to the form of a polynomial ψ in both variables (rather than in just one as in [1]) was asked of me by Felix Browder.

Reference

1. N. Levinson, *A canonical form for an analytic function of several variables at a critical point*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 68–69.

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