THE GENERALIZED POINCARÉ CONJECTURE IN HIGHER DIMENSIONS

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The Poincaré conjecture says that every simply connected closed 3-manifold is homeomorphic to the 3-sphere S^3. This has never been proved or disproved. The problem of showing whether every closed simply connected n-manifold which has the homology groups of S^n, or equivalently is a homotopy sphere, is homeomorphic to S^n, has been called the generalized Poincaré conjecture.

We prove the following theorem.

THEOREM A. *If M^n is a closed differentiable (C^∞) manifold which is a homotopy sphere, and if $n \neq 3$, 4, then M^n is homeomorphic to S^n.*

We would expect that our methods will yield Theorem A for combinatorial manifolds as well, but this has not been done.

The complete proof will be given elsewhere. Here we give an outline of the proof and mention other related and more general results.

The first step in the proof is the construction of a nice cellular type structure on any closed C^∞ manifold M. More precisely, define a real valued f on M to be a *nice* function if it possesses only nondegenerate critical points and for each critical point β, $f(\beta) = \lambda(\beta)$, the index of β.

THEOREM B. *On every closed C^∞ manifold there exist nice functions.*

The proof of Theorem B is begun in our article [3]. In the terminology of [3], it is proved that a gradient system can be C^1 approximated by a system with stable and unstable manifolds having normal intersection with each other. This is the announced Theorem 1.2 of [3]. From this approximation we are then able to construct the function of Theorem B.

The stable manifolds of the critical points of a nice function can be thought of as cells of a complex while the unstable manifolds are the duals. This structure has the advantage over previous structures that both the cells and the duals are differentiably imbedded in M. We believe in fact that nice functions will replace much of the use of C^1 triangulations and combinatorial methods in differential topology.

In our work here we do not actually use the cells themselves as much as the neighborhoods of the skeletons of this structure. More precisely, if f is a nice function on M, let $X_k = f^{-1}[0, k+1/2]$.

Then each X_k is a compact C^∞ manifold with boundary and one can think of X_k as being obtained from X_{k-1} by attaching a number of k-cells and "thickening" them.

If M is a manifold with boundary, D a disk, $f: \partial D \to \partial M$ a differentiable imbedding with a trivial normal bundle B and γ a homotopy class of cross-sections $\partial D \to B$, then a new manifold $M \cup D = M \cup_{f,\gamma} D$ is defined. We do not define $M \cup D$ here but only remark that $M \cup D$ is roughly M with D attached by f, thickened according to γ, and smoothed.

Then it can be shown that $X_k = X_{k-1} \cup D_1^k \cup \cdots \cup D_r^k$, where $\dim D_i^k = k$ (D, D^k, etc. always denote disks).

The *handlebodies* are defined to be all manifolds of the form $D^n \cup D_1^s \cup \cdots \cup D_k^s$. Fixing n, k, s, the set of all such manifolds is denoted by $\mathcal{3C}(n, k, s)$.

For our main theorems we prove that under homotopy assumptions, the X_k are handlebodies. The following is a major step.

THEOREM C (THE HANDLEBODY THEOREM). *Let* $n \geq \max(2s+2, 5)$, $H \in \mathcal{3C}(n, k, s)$, $K = H \cup D_1 \cup \cdots \cup D_r$, $\dim D_i = s+1$ *and* $\pi_s(K) = 0$. *Also assume* $\pi_1(H \cup D_1 \cup \cdots \cup D_{r-k}) = 1$ *if* $s = 1$. *Then* $K \in \mathcal{3C}(n, r-k, s+1)$.

We shall not try to summarize the proof of Theorem C.

Using Theorems B and C and the previous consideration one can obtain the following theorem.

THEOREM D. *Let* M^n *be a closed* C^∞ *manifold which is* $(m-1)$-*connected and* $n \geq \max(2m, 5)$. *Then there is a nice function* f *on* M *with type numbers satisfying*

$$M_0 = M_n = 1 \quad \text{and} \quad M_i = 0 \quad \text{for} \quad 0 < i < m \quad \text{and} \quad n - m < i < n.$$

Special cases of Theorem D are first Theorem E if $n = 2m+1$, and Theorem F if $n = 2m$.

THEOREM E. *Let* M *be a closed* $(m-1)$-*connected closed* C^∞ $(2m+1)$-*manifold. Then* $M = H \cup H'$ *where* H, $H' \in \mathcal{3C}(2m+1, k, m)$.

This generalizes the Heegard decomposition of a 3-manifold.

THEOREM F. *Let* M *be a closed* $(m-1)$-*connected* C^∞ $2m$-*manifold*, $m \neq 2$. *Then there is a nice function on* M *whose type numbers equal the Betti numbers of* M.

Theorem F implies the even dimensional part of Theorem A.

According to Thom (see [2]) two closed oriented C^∞ manifolds M_1 and M_2 are *J-equivalent* if there exists a C^∞ manifold X with boundary $M_1 - M_2$ and each M_i is a deformation retract of X. Using methods similar to the preceding ones we are able to prove the following:

THEOREM G. *If two homotopy spheres M_1 and M_2 of dimension $2m - 1$, $m \neq 2$ are J-equivalent, then they are diffeomorphic.*

Using Mazur's Theorem [1], the part of Theorem A for odd dimensional manifolds follows from Theorem G.

THEOREM H. *There exists a triangulated manifold with no differentiable structure at all.*

This follows from Theorem G and work of Milnor [2]. Take his manifold of W_0 of Theorem 4.1 of [2] for $k = 3$ and attach a $4k$ cell. (In our context W_0 can be viewed as a certain handlebody in $\mathcal{3C}(4k, 8, 2k)$.) The following also follows from Milnor [2] and Theorem G:

THEOREM I. *The groups Γ^{2m+1} are finite, for all m.*

For example, for $\Gamma^5 = 0$, there are precisely 28 differentiable structures on S^7, etc.

REFERENCES

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3. S. Smale, *Morse inequalities for a dynamical system*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 43–49.

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