EXTENSION OF CONTINUOUS FUNCTIONS IN βN

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1. **Introduction.** The present considerations arose from the following problem: let $p \in \beta N - N$; is $\beta N - N - \{p\}$ C^*-embedded in $\beta N - N$—i.e., is $\beta(\beta N - N - \{p\})$ equal to $\beta N - N$?[3] We prove, assuming the continuum hypothesis (designated by [CH]), that the answer is negative. More generally, see Theorem 4.6. The corresponding question for βD, where D is any discrete space, is discussed in §5. The proofs depend upon results about F-spaces. We also prove [CH] that all open subsets of $\beta R - R$ are F-spaces and that all open subsets of $\beta N - N$ are zero-dimensional F-spaces.

2. **Background.** All spaces considered are assumed to be completely regular. N is the countably infinite discrete space, R the space of reals. $C^*(X)$ denotes the ring of all bounded continuous functions from X into R. A *zero-set* in X is the set $Z(f)$ of zeros of a continuous function f. A *cozero-set* is the complement of a zero-set. Countable unions of cozero-sets are cozero-sets. A subspace S of X is *C^*-embedded* in X if every function in $C^*(S)$ has a continuous extension to all of X. βX denotes the Stone-Čech compactification of X, i.e., a compactification of X in which X is C^*-embedded.

The main results depend upon properties of F-spaces. Each of the following conditions characterizes X as an F-space: *every cozero-set in X is C^*-embedded*; *any two disjoint cozero-sets are completely separated in X* (i.e., some function in $C^*(X)$ is equal to 0 on one of them and 1 on the other). Let

(2.1) $K = \beta Y - Y$, *where Y is locally compact and σ-compact but not compact.*

Then K is a compact F-space without isolated points, and $|K| \geqq \exp \exp \aleph_0$. Examples: $K = \beta N - N$, $K = \beta R - R$. For the algebraic significance of F-spaces, as well as for proofs of quoted results, see [1] and [2, Chapter 14].

The following two properties of a space X are equivalent: any two completely separated sets are contained in complementary open-and-closed sets; βX is totally disconnected. We express these conditions by saying that X is *zero-dimensional*. (For the requisite defini-

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[3] Symbols and terms are defined in §2. For additional details, see [2].

tion of dimension, see [**2**, Chapter 16].) If X is zero-dimensional, so is any C^*-embedded subspace.

3. **Preliminary results.** For any space Y, vY denotes the set of all points p of βY such that every zero-set in βY that contains p also meets Y. When $vY = Y$, Y is said to be *realcompact*. All σ-compact spaces are realcompact [**2**, Chapter 8].

3.1. LEMMA. *If Y is locally compact and realcompact, then each zero-set in $\beta Y - Y$ is the closure of its interior.*

REMARK. It suffices to prove that a nonempty zero-set Z has nonempty interior—for if $p \in Z - \operatorname{cl} \operatorname{int} Z$, then some zero-set Z' disjoint from $\operatorname{int} Z$ contains p, and $\operatorname{int}(Z \cap Z')$ is empty.

PROOF. Since Y is locally compact, $\beta Y - Y$ is compact and is therefore C^*-embedded in βY; hence $Z = \boldsymbol{Z}(f) - Y$ for some $f \in C^*(\beta Y)$. Let $p \in Z$. Since $p \notin vY$, there is a function g in $C^*(\beta Y)$ that vanishes at p but nowhere on Y. Define $h = |f| + |g|$; then $p \in \boldsymbol{Z}(h) \subset Z$. Let (y_n) be a sequence of distinct points in Y on which h approaches 0. Choose disjoint compact neighborhoods V_n of y_n such that $|h(y) - h(y_n)| < 1/n$ for $y \in V_n$. It is easy to see that there exists a function $u \in C^*(\beta Y)$ that is equal to 1 at each y_n and equal to 0 everywhere on $Y - \bigcup_n V_n$. If q is any point of $\beta Y - Y$ at which $u(q) \neq 0$, then every neighborhood of q meets infinitely many of the compact sets V_n; hence $h(q) = 0$. Thus $\boldsymbol{Z}(h)$ contains the nonvoid open subset $\beta Y - Y - \boldsymbol{Z}(u)$ of $\beta Y - Y$.

Local compactness is critical: an easy example shows that the conclusion of the lemma fails for $\beta \boldsymbol{Q} - \boldsymbol{Q}$ ($\boldsymbol{Q}$ = space of rationals).

3.2. REMARK. *If Y is locally compact and realcompact, but not compact, then $\beta Y - Y$ is not basically disconnected*—for 3.1 would imply that it is a P-space. (See [**2**] for definitions and for other proofs for $\beta\mathbf{N} - \mathbf{N}$.)

3.3. Given X, let $S \subset X$ and $p \in X - S$. The main results will be formulated in terms of the condition

(p, S): *There exist a neighborhood V of p and a cozero-set $H \subset S$ such that $S \cap V - H$ has void interior.*

Trivially, (p, S) holds whenever $p \notin \operatorname{cl} S$; and if S is a cozero-set, then (p, S) holds for every $p \notin S$.

3.4. LEMMA. *Let F be a compact set in X such that $(p, X - F)$ holds for all $p \in F$. Then* $\operatorname{int} Z \subset F \subset Z$ *for some zero-set Z.*

PROOF. There exist a finite open cover $\{V_1, \cdots, V_n\}$ of F and zero-sets $Z_1, \cdots, Z_n$ containing F such that $\operatorname{int} Z_k - F \subset X - V_k$. Let

$Z_0 = \bigcap_k Z_k$; then Z_0 is a zero-set containing F and $\operatorname{int} Z_0 - F \subset X - \bigcup_k V_k \subset X - F$, so that $\operatorname{cl}(\operatorname{int} Z_0 - F)$ does not meet F. Since F is compact, it is contained in a zero-set Z' disjoint from $\operatorname{int} Z_0 - F$ (see, e.g., [**2**, 3.11]). This implies that $\operatorname{int}(Z_0 \cap Z') \subset F \subset Z_0 \cap Z'$.

3.5. THEOREM. *Let X be an F-space, and let $S \subset X$ and $p \in \operatorname{cl} S - S$. If $S \cap V$ is open for some neighborhood V of p, and if (p, S) holds, then S is C^*-embedded in $S \cup \{p\}$.*

PROOF. There exist a neighborhood V of p and a cozero-set $H \subset S$ such that $S \cap V$ is open and is disjoint from $\operatorname{int}(X - H)$. Given $f \in C^*(S)$, let $g \in C^*(H \cup \{p\})$ be an extension of $f|H$ (see §2). Define h on $S \cup \{p\}$ to agree with f on S and with g at p. Since $H \cap V$ is dense in $(S \cup \{p\}) \cap V$, h is a continuous extension of f.

4. The main results.

4.1. THEOREM. *Let X be an F-space and let $S \subset X$ be a union of $\aleph_1$ cozero-sets S_α (in X). Then* (a) *S is an F-space*; (b) *if X is zero-dimensional, so is S*; (c) *if $G \subset S$ and $G \cap S_\alpha$ is a cozero-set in S (for each α), then G is C^*-embedded in S.*

PROOF. (c). We may assume that $S = \bigcup_{\alpha < \omega_1} S_\alpha$ and that $S_0 \subset S_1 \subset \cdots$. Notice that every S_ξ is an F-space. Let $g \in C^*(G)$ be given. Put $g_\xi = g | G \cap S_\xi$. Given $\alpha < \omega_1$, assume that g_ξ has been extended to $s_\xi \in C^*(S_\xi)$, for each $\xi < \alpha$, and that $s_0 \subset s_1 \subset \cdots$. The function $\bigcup_{\xi < \alpha} s_\xi \cup g_\alpha$ is well defined and continuous on the cozero-set $\bigcup_{\xi < \alpha} S_\xi \cup (G \cap S_\alpha)$ in the F-space S_α; hence it has an extension to a function $s_\alpha \in C^*(S_\alpha)$. Finally, $\bigcup_{\alpha < \omega_1} s_\alpha$ is a continuous extension of g to all of S.

(a) If G is a cozero-set in S, then by (c), G is C^*-embedded.

(b) Completely separated sets in S are contained in disjoint cozero-sets A and B in S. Let $g \in C^*(A \cup B)$ be equal to 0 on A and to 1 on B. Note that every S_α is zero-dimensional. In the proof of (c), (with $G = A \cup B$), add to the induction hypothesis that $\bigcup_{\xi < \alpha} s_\xi$ is two-valued; then s_α may be taken to be two-valued.

4.2. COROLLARY. [CH]. *All open subsets of $\beta\mathbf{R} - \mathbf{R}$ are F-spaces. All open subsets of $\beta\mathbf{N} - \mathbf{N}$ are zero-dimensional F-spaces.*

PROOF. Both $\beta\mathbf{R} - \mathbf{R}$ and $\beta\mathbf{N} - \mathbf{N}$ have just $\exp \aleph_0$ zero-sets.

4.3. THEOREM. [CH]. *Let X be an F-space having just* $\exp \aleph_0$ *zero-sets. Let S be open and let $p \in \operatorname{cl} S - S$, and suppose that (p, S) fails. Then* (a) *S is not C^*-embedded in $S \cup \{p\}$*; (b) $|\beta S - S| \geqq \exp \exp \aleph_1$; (c) *if X is zero-dimensional, there is a two-valued function in $C^*(S)$ that has no continuous extension to p.*

PROOF. (a). Let $(S_\xi)_{\xi < \omega_1}$ be a family of cozero-sets in X whose union is S, and let $(V_\xi)_{\xi<\omega_1}$ be a base of zero-set-neighborhoods of p. Inductively, for each $\alpha<\omega_1$, assume that cozero-sets A_ξ and B_ξ, contained in S, have been defined for all $\xi<\alpha$. Because (p, S) fails, we can choose disjoint, nonempty cozero-sets A_α and B_α contained in

$$S\cap V_\alpha - \bigcup_{\xi<\alpha} (A_\xi \cup B_\xi \cup S_\xi).$$

Define $A=\bigcup_{\alpha<\omega_1} A_\alpha$, $B=\bigcup_{\alpha<\omega_1} B_\alpha$, and $G=A\cup B$. By construction, for each $\xi<\omega_1$, $G\cap S_\xi$ is the cozero-set $\bigcup_{\alpha\leq\xi}(A_\alpha\cup B_\alpha)\cap S_\xi$. By 4.1(c), G is C^*-embedded in S. But A and B are complementary open sets in G and each meets every neighborhood of p; therefore G is not C^*-embedded in $G\cup\{p\}$. It follows that S is not C^*-embedded in $S\cup\{p\}$.

(c) This now follows from 4.1(b).

(b) Since $|S| \leqq \exp\exp\aleph_0$ (every point being an intersection of zero-sets), it is sufficient to show that $|\beta S|\geqq \exp\exp\aleph_1$. Because G is C^*-embedded in S, $|\beta S|\geqq|\beta G|$. Clearly, G contains a C^*-embedded copy of the discrete space D of cardinal $\aleph_1$; so $|\beta G|\geqq|\beta D|$. Finally, $|\beta D|=\exp\exp\aleph_1$, as is well known.

4.4. Corollary [CH]. *Let X be an F-space with just* $\exp\aleph_0$ *zero-sets, and let $S\subset X$ be open and $p\in \text{cl } S-S$. Then S is C^*-embedded in $S\cup\{p\}$ if and only if (p, S) holds.*

Proof. 3.5 and 4.3(a).

4.5. Question. Suppose that X is zero-dimensional and that a dense subset S of X is not C^*-embedded in X; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to X? It is easy to see that the answer is "yes" in case S itself is zero-dimensional.

4.6. Theorem [CH]. *Let K be a compact F-space of the form* (2.1) *that has just* $\exp\aleph_0$ *zero-sets.* (*E.g.*, $K=\beta\mathbf{N}-\mathbf{N}$ *or* $K=\beta\mathbf{R}-\mathbf{R}$.) *Then*:

(a) *No proper dense subset is C^*-embedded—i.e., the equation $\beta X=K$ has the unique solution $X=K$.*

(b) *The following are equivalent for an open set S*:

(i) *S is C^*-embedded in K.*

(ii) *S is a cozero-set.*

(iii) *(p, S) holds for all $p\in K-S$.*

(c) *If S is open but is not a cozero-set, then* $|\beta S-S|\geqq\exp\exp\aleph_1$.

(d) *If K is totally disconnected* (*e.g.*, $K=\beta\mathbf{N}-\mathbf{N}$), *and if S is open but is not a cozero-set, then there is a two-valued function in $C^*(S)$ that has no continuous extension to all of K.*

Proof. We prove first that (iii) implies (ii): by 3.1, cl int $Z=Z$ for every zero-set Z in K; hence (iii) and 3.4 imply that $K-S$ is a zero-set. Conclusions (b), (c), and (d) now follow from 4.3. Since no point of K is isolated, 3.1 shows that no point is a zero-set; by (b), the complement of a point is not C^*-embedded, and this implies (a).

4.7. REMARK. *Let X be a compact F-space; if $S \subset X$ and $|X-S| <\exp\exp \aleph_0$, then S is pseudocompact* (i.e., every continuous function is bounded). For if S admits an unbounded function, then S contains $\mathbf{N}$ as a closed subset. Now, $\mathbf{N}$ is C^*-embedded in X [2, 14N] and so $\mathrm{cl}_X \mathbf{N}=\beta\mathbf{N}$. But $X-S\supset\beta\mathbf{N}-\mathbf{N}$ and $|\beta\mathbf{N}-\mathbf{N}|=\exp\exp \aleph_0$.

5. **The space $\beta D-D$ for (uncountable) discrete D.** If $A\subset D$ and $p\in \mathrm{cl}_{\beta D} A-D$, then $\mathrm{cl}\, A-D$ is an open-and-closed neighborhood of p in $\beta D-D$; these sets form a base at p in $\beta D-D$. Let E_0 be the set of points in $\beta D-D$ in the closures of countable subsets of D, $E=\beta D-D-E_0$, E_1 the set of points of E in the closures of subsets of D of cardinal $\aleph_1$. Then E_0 is countably compact and is open and dense in $\beta D-D$. Every compact subset of E_0 has an open neighborhood in E_0 homeomorphic with $\beta\mathbf{N}-\mathbf{N}$. Since D is an F-space, so are βD and its compact subspace E.

5.1. THEOREM [CH]. *If $p\in E_0$, then $\beta D-D-\{p\}$ is not C^*-embedded in $\beta D-D$.*

PROOF. p has an open neighborhood in E_0 homeomorphic with $\beta\mathbf{N}-\mathbf{N}$, and 4.6(a) applies locally.

5.2. THEOREM. *If S is an open subset of E_0, then either S has compact closure in E_0 or S has infinitely many limit points in E_1.*

PROOF. Let $\mathfrak{N}$ be a maximal family of disjoint, countably infinite subsets N of D for which $\mathrm{cl}\, N-D\subset S$. Since S is open, $\mathrm{cl}\bigcup\mathfrak{N}\supset S$. If $\mathfrak{N}$ is countable, then $\mathrm{cl}\bigcup\mathfrak{N}-D\subset E_0$. If $\mathfrak{N}$ is uncountable, it has a subfamily $\mathfrak{N}'$ of cardinal $\aleph_1$. Let $\mathfrak{F}$ be the filter on D of all sets that contain all but finitely many points of N for all but countably many $N\in\mathfrak{N}'$. Clearly, $\mathfrak{F}$ is contained in infinitely many (in fact, $\exp\exp \aleph_1$) ultrafilters $\mathfrak{U}$. For each such $\mathfrak{U}$, consider $p=\lim \mathfrak{U}$. Because the members of $\mathfrak{N}'$ are disjoint, every member of $\mathfrak{U}$ is uncountable; hence $p\in E$. Since $\mathfrak{U}$ contains the set $\bigcup\mathfrak{N}'$ of cardinal $\aleph_1$, $p\in E_1$.

5.3. LEMMA (HENRIKSEN). *If $p\in E-vD$,[4] then $E-\{p\}$ is C^*-embedded in E.*

PROOF.[5] Since $p\notin vD$, some function $f\in C^*(\beta D)$ vanishes at p but

[4] It is known that if $|D|$ is smaller than the first strongly inaccessible cardinal, then $vD=D$ (see §3).

[5] Due to Henriksen and Jerison; Henriksen's original proof was based on some results in the theory of lattice-ordered rings.

nowhere on D. Every neighborhood of a point of E meets D in an uncountable set on which $|f|$ is bounded away from zero. Hence $E-\boldsymbol{Z}(f)$ is a dense cozero-set in the F-space E, and therefore the intermediate subspace $E-\{p\}$ is C^*-embedded in E.

5.4. THEOREM (ISBELL-JERISON). *If $p\in E-vD$,[4] then $\beta D-D-\{p\}$ is C^*-embedded in $\beta D-D$.*

PROOF. Given $g\in C^*(\beta D-D-\{p\})$, consider its restriction $f=g|E-\{p\}$. By Henriksen's lemma, f can be extended continuously to p—say with the value 0 at p. It suffices to show that $|g|$ stays small near p. Given $\epsilon>0$, let V be an open-and-closed neighborhood of p such that $|f(q)|<\epsilon$ for all $q\in V\cap E$. Let S be the set of all points $x\in V\cap E_0$ such that $|g(x)|>\epsilon$; then cl S meets E in at most the single point p. To show that $p\notin$ cl S, one may observe[6] that S is open and apply 5.2. Thus, $|g(x)|\leqq\epsilon$ on the neighborhood $V-$cl S of p.

5.5. QUESTION. Is E_0 C^*-embedded in $\beta D-D$? If so, then E_0 is a zero-dimensional F-space. Note that 4.1 and [CH] yield the latter for the case $|D|=\aleph_1$. If E_0 is not C^*-embedded in $\beta D-D$, then it is not C^*-embedded in $D\cup E_0$; this would imply that $D\cup E_0$ is not a normal space. In the case $|D|=\aleph_1$ (at least), it would also imply, by 4.5 and [CH], that some two-valued function in $C^*(E_0)$ cannot be extended continuously to βD.

We remark that the problem of extending two-valued functions from E_0 (for arbitrary D) can be formulated in the following way. Let $\mathcal{P}$ be the Boolean algebra of all subsets of D, $\mathcal{C}$ the subring of all countable subsets, and $\mathfrak{F}$ the ideal of all finite subsets. Let Λ be the set of all endomorphisms λ of $\mathcal{C}/\mathfrak{F}$ that satisfy (i): $\lambda(x)\subset x$, and (ii): $\lambda(\lambda(x))=\lambda(x)$. Then the following are equivalent: every two-valued function in $C^*(E_0)$ has a continuous extension to all of βD; every $\lambda\in\Lambda$ can be extended to $\mathcal{P}/\mathfrak{F}$ so as to satisfy (i) and (ii).

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[6] This is a modification of the Isbell-Jerison argument.