ON DIFFERENTIABLE FUNCTIONS

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1. Let $f(x)$ be a real function defined over the closed interval $[a, b]$ and differentiable at each point of the interval. The following is a well-known classical theorem.

DARBOUX'S THEOREM. *If* $\xi, \eta \in [a, b]$ *and* $f'(\xi) = c$, $f'(\eta) = d$ *and* $c < d$, *then, if* $c < k < d$, *there exists at least one point* ζ *lying between* ξ *and* η *such that* $f'(\zeta) = k$.

Recently in 1947, Clarkson [1] developed the above property into the following important theorem concerning the behavior of the derivative $f'(x)$.

CLARKSON'S THEOREM. *Let* α, β *with* $\alpha < \beta$ *be any two real numbers and let us denote the aggregate of points in* $[a, b]$ *such that* $\alpha < f'(x) < \beta$ *by*

$$E_{\alpha\beta} = E(x; \alpha < f'(x) < \beta);$$

then $E_{\alpha\beta}$ *is either void or* $\mathfrak{M}(E_{\alpha\beta}) > 0$, *where* $\mathfrak{M}(E)$ *is the Lebesgue measure of the set* E.

2. In this note, we intend to give a more detailed description of $E_{\alpha\beta}$. We prove the following

THEOREM. *The set* $E_{\alpha\beta}$ *gives rise to a set of nonoverlapping and nonabutting open sub-intervals* $\{I_i\}$ *in the space* $[a, b]$ *and a closed set* G, *which is the complementary set of* $\{I_i\}$ *with respect to* $[a, b]$, *such that* $E_{\alpha\beta}$ *is void in each* I_i *and metrically dense everywhere in* G.

3. We first show that $E_{\alpha\beta}$ is metrically dense in itself. For, take any point P of $E_{\alpha\beta}$ and any neighborhood U_P containing P as its interior point, since $U_P \cap E_{\alpha\beta}$ is not void, from Clarkson's theorem, $\mathfrak{M}(U_P \cap E_{\alpha\beta}) > 0$. Thus, each point of $E_{\alpha\beta}$ is a limiting point of the set. I.e., $E_{\alpha\beta} \subseteq E'_{\alpha\beta}$. Moreover, for each point $P \in CE'_{\alpha\beta}$, it is always possible for us to construct a largest open sub-interval I_P relative to the space $[a, b]$ such that it contains P as its interior point and has its two end points belonging to the set $E'_{\alpha\beta}$,[1] and such that all of its interior points are the points of $CE'_{\alpha\beta}$. Thus, $E_{\alpha\beta}$ is void in each I_P, since $E_{\alpha\beta} \subseteq E'_{\alpha\beta}$. Hence, we see that, corresponding to the set

[1] In case $a \in CE'_{\alpha\beta}$ (similarly for b), the corresponding U_a takes a as one of its end points and the other end point of U_a is of course a point of $E'_{\alpha\beta}$.

$CE'_{\alpha\beta}$, there exists a set of open sub-intervals $\{I_P\}$, in each of which $E_{\alpha\beta}$ is void. Moreover, for any two different points P and P' of $CE'_{\alpha\beta}$, we have $I_P \equiv I_{P'}$ or $I_P \cap I_{P'} = 0$. For, otherwise, there should be an interior point of I_P (or $I_{P'}$), which is also a point of the set $E'_{\alpha\beta}$. This contradicts the mode of construction for $\{I_P\}$. Further, by the same reason, I_P and $I_{P'}$ can not be abutting. Therefore, the set of the open sub-intervals $\{I_P\}$ corresponding to the set $CE'_{\alpha\beta}$ is a countable set of nonoverlapping and nonabutting open sub-intervals $\{I_i\}$ in the space $[a, b]$. Let the complementary set of $\{I_i\}$ with respect to $[a, b]$ be G. Then G is closed. And $E_{\alpha\beta}$ is metrically dense everywhere in G, since, for any point $Q \in G$, $\mathfrak{M}(U_Q \cap E_{\alpha\beta}) > 0$ is satisfied for any arbitrary neighborhood of Q. This proves the theorem.

Reference

1. J. A. Clarkson, *A property of derivatives*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 124–125.

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A q-BINOMIAL COEFFICIENT SERIES TRANSFORM

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Following a standard notation [**1**; **4**; **5**] we define the q-binomial coefficients by means of

$$\begin{bmatrix} x \\ n \end{bmatrix} = \prod_{j=1}^{n} \frac{q^{x-j+1} - 1}{q^j - 1}, \tag{1}$$

or sometimes more conveniently by the notation

$$\begin{bmatrix} x \\ n \end{bmatrix} = [x]_n/[n]!, \tag{2}$$

where

$$\begin{gathered} [x]_n = [x][x-1] \cdots [x-n+1], \\ [x] = (q^x - 1)/(q - 1), \\ [n]! = [n]_n, \qquad [0]! = [x]_0 = 1, \qquad \begin{bmatrix} x \\ 0 \end{bmatrix} = 1. \end{gathered} \tag{3}$$