$CE'_{\alpha\beta}$, there exists a set of open sub-intervals $\{I_P\}$, in each of which $E_{\alpha\beta}$ is void. Moreover, for any two different points P and P' of $CE'_{\alpha\beta}$, we have $I_P \equiv I_{P'}$ or $I_P \cap I_{P'} = 0$. For, otherwise, there should be an interior point of I_P (or $I_{P'}$), which is also a point of the set $E'_{\alpha\beta}$. This contradicts the mode of construction for $\{I_P\}$. Further, by the same reason, I_P and $I_{P'}$ can not be abutting. Therefore, the set of the open sub-intervals $\{I_P\}$ corresponding to the set $CE'_{\alpha\beta}$ is a countable set of nonoverlapping and nonabutting open sub-intervals $\{I_i\}$ in the space $[a, b]$. Let the complementary set of $\{I_i\}$ with respect to $[a, b]$ be G. Then G is closed. And $E_{\alpha\beta}$ is metrically dense everywhere in G, since, for any point $Q \in G$, $\mathfrak{M}(U_Q \cap E_{\alpha\beta}) > 0$ is satisfied for any arbitrary neighborhood of Q. This proves the theorem.

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A q-BINOMIAL COEFFICIENT SERIES TRANSFORM

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Following a standard notation [1; 4; 5] we define the q-binomial coefficients by means of

$$\begin{bmatrix} x \\ n \end{bmatrix} = \prod_{j=1}^{n} \frac{q^{x-j+1} - 1}{q^j - 1}, \tag{1}$$

or sometimes more conveniently by the notation

$$\begin{bmatrix} x \\ n \end{bmatrix} = [x]_n/[n]!, \tag{2}$$

where

$$\begin{aligned} [x]_n &= [x][x-1] \cdots [x-n+1], \\ [x] &= (q^x - 1)/(q-1), \end{aligned} \tag{3}$$

$$[n]! = [n]_n, \qquad [0]! = [x]_0 = 1, \qquad \begin{bmatrix} x \\ 0 \end{bmatrix} = 1.$$

We also note the properties

$$\left[\begin{matrix} n \\ k \end{matrix}\right] = \left[\begin{matrix} n \\ n-k \end{matrix}\right] \quad (0 \leqq k \leqq n), \qquad \left[\begin{matrix} n \\ k \end{matrix}\right] = 0 \qquad (n < k), \tag{4}$$

and the fact that the ordinary binomial coefficients arise as a limiting case:

$$\binom{x}{n} = \lim_{q \to 1} \left[\begin{matrix} x \\ n \end{matrix}\right]. \tag{5}$$

We define now what we shall call a *q-Vandermonde transform*:

$$F(n) = q^{n(n-1)/2} \sum_{k=0}^{n} (-1)^k \left[\begin{matrix} n \\ k \end{matrix}\right] \left[\begin{matrix} a+bk \\ n \end{matrix}\right] q^{k(k-n)} f(k). \tag{6}$$

The purpose of this paper is to give a brief formal proof of the following result.

THEOREM. *Let a and b be non-negative integers. In* (6), *let $f(k)$ be independent of n, and $f(0) = 1$. Then*

$$\sum_{n=0}^{\infty} (-1)^n F(n) u^n = \sum_{k=0}^{\infty} \left[\begin{matrix} a+bk \\ k \end{matrix}\right] q^{k(k-1)/2} u^k f(k) \prod_{j=0}^{a+bk-k-1} (1 - uq^j). \tag{7}$$

PROOF. We remark in the first place that if x is a non-negative integer, then

$$\left[\begin{matrix} x \\ n \end{matrix}\right] = \frac{[x]!}{[n]![x-n]!}, \tag{8}$$

from which we obtain

$$\left[\begin{matrix} n \\ k \end{matrix}\right] \left[\begin{matrix} x \\ n \end{matrix}\right] = \left[\begin{matrix} x \\ k \end{matrix}\right] \left[\begin{matrix} x-k \\ n-k \end{matrix}\right]. \tag{9}$$

In the second place, we note the known q-identity

$$\sum_{n=0}^{m} (-1)^n \left[\begin{matrix} m \\ n \end{matrix}\right] q^{n(n-1)/2} u^n = \prod_{j=0}^{m-1} (1 - uq^j). \tag{10}$$

Now, from (6), we find

$$\left[\begin{matrix} a+bn \\ n \end{matrix}\right] q^{n(n-1)/2} f(n) - (-1)^n F(n)$$
$$= \sum_{k=0}^{n-1} (-1)^{k+n-1} \left[\begin{matrix} n \\ k \end{matrix}\right] \left[\begin{matrix} a+bk \\ n \end{matrix}\right] q^{k(k-1)/2} q^{(n-k)(n-k-1)/2} f(k),$$

where we have made use of the simple identity

$$\frac{1}{2}k(k-1)+\frac{1}{2}(n-k)(n-k-1)=\frac{1}{2}n(n-1)+k(k-n).$$

Multiplying each side by u^n and summing we next find

$$\begin{aligned}
&\sum_{n=0}^{\infty}\begin{bmatrix} a+bn \\ n \end{bmatrix} q^{n(n-1)/2}f(n)u^n-\sum_{n=0}^{\infty}(-1)^nF(n)u^n \\
&=\sum_{n=1}^{\infty}u^n\sum_{k=0}^{n-1}(-1)^{k+n-1}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} a+bk \\ n \end{bmatrix} q^{k(k-1)/2}q^{(n-k)(n-k-1)/2}f(k) \\
&=\sum_{n=0}^{\infty}u^{n+1}\sum_{k=0}^{n}(-1)^{k+n}\begin{bmatrix} n+1 \\ k \end{bmatrix}\begin{bmatrix} a+bk \\ n+1 \end{bmatrix} q^{k(k-1)/2}q^{(n-k)(n-k+1)/2}f(k) \\
&=\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2}f(k) \\
&\quad\cdot\sum_{n=k}^{\infty}(-1)^n\begin{bmatrix} a+bk-k \\ n+1-k \end{bmatrix} q^{(n-k)(n-k+1)/2}u^{n+1} \\
&=\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2}f(k)\sum_{n=1}^{\infty}(-1)^{n+k-1}\begin{bmatrix} a+bk-k \\ n \end{bmatrix} q^{n(n-1)/2}u^{n+k} \\
&=-\sum_{k=0}^{\infty}\begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2}f(k)u^k\sum_{n=0}^{\infty}(-1)^n\begin{bmatrix} a+bk-k \\ n \end{bmatrix} q^{n(n-1)/2}u^n \\
&\quad+\sum_{k=0}^{\infty}\begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2}f(k)u^k.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^nF(n)u^n \\
&=\sum_{k=0}^{\infty}\begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2}f(k)u^k\sum_{n=0}^{\infty}(-1)^n\begin{bmatrix} a+bk-k \\ n \end{bmatrix} q^{n(n-1)/2}u^n.
\end{aligned}\tag{11}$$

But now, by (4), and assuming $a+bk-k$ to be a non-negative integer, we see that the inner summation is finite, and so by relation (10) the desired result follows.

In the special case when $q\to 1$ we obtain

COROLLARY. *If*

$$F(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{n}f(k),\tag{12}$$

and $f(k)$ *is independent of* n *and* $f(0)=1$, *then*

$$(13) \qquad \sum_{k=0}^{\infty} \binom{a+bk}{k} z^k f(k) = x^a \sum_{n=0}^{\infty} (-1)^n F(n) u^n,$$

where $z=(x-1)/x^b$ *and* $u=(x-1)/x$.

From this corollary it is possible to deduce some interesting consequences, and these will be discussed in detail elsewhere. For example it is easy to prove the inversion

THEOREM. *If* $F(n)$ *is defined by* (12), *then*

$$(14) \quad \binom{a+bn}{n} f(n) = \sum_{k=0}^{n} (-1)^k \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} F(k).$$

Relation (13) also leads to the formula

$$(15) \sum_{k=0}^{\infty} \frac{w}{w+bk} \binom{a+bk}{k} z^k = x^a \sum_{n=0}^{\infty} (-1)^n \binom{a-w}{n} \binom{w/b+n}{n}^{-1} u^n.$$

At the time of writing this paper it is not known whether a formula similar to (15) follows for the q-series; however, we might expect to choose something like $f(k) = [w]/[w+bk]$ and apply (7) together with known finite formulas for q-series to evaluate (6). It is hoped to discuss this elsewhere.

Finally we note that in analogy to the Vandermonde transform (12) we may define an Abel transform and we obtain the following

THEOREM. *If*

$$(16) \qquad F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+bk)^n}{n!} f(k),$$

where $f(k)$ *is independent of* n *and* $f(0)=1$, *then*

$$(17) \qquad \sum_{k=0}^{\infty} \frac{(a+bk)^k}{k!} z^k f(k) = x^a \sum_{n=0}^{\infty} (-1)^n F(n) u^n,$$

where $z=u/x^b$ *and* $u=\log x$.

The corresponding inverse to (16) is slightly different from (14) and reads as follows:

THEOREM. *If* $F(n)$ *is defined by* (16), *then*

$$(18) \qquad \frac{(a+bn)^n}{n!} f(n) = \sum_{k=0}^{n} (-1)^k F(k) \frac{(a+bn)^{n-k}}{(n-k)!} \cdot \frac{a+bk}{a+bn} \cdot$$

In analogy to (15) we then find

$$(19)\quad \sum_{k=0}^{\infty} \frac{w}{w+bk}\cdot\frac{(a+bk)^k}{k!}z^k = x^a \sum_{n=0}^{\infty} (-1)^n \frac{(a-w)^n}{n!}\binom{w/b+n}{n}^{-1} u^n.$$

Proofs of these results will be given elsewhere together with applications in finding convolution identities generalizing those discussed in [**3**].

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