ON A PROPERTY OF POSITIVE-DEFINITE FUNCTIONS[1]

BY R. G. LAHA

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In the present note we prove the following theorem:

THEOREM. *Let* $\{t_k\}$ $(k=\pm 1, \pm 2, \cdots)$ *be a sequence of real numbers such that* $t_k>0$ *and* $t_{-k}=-t_k$ *for any* $k>0$ *and that* $t_k\to 0$ *as* $k\to\infty$. *Let* $f(t)$ *be a positive-definite function of the real variable* t *and let* $\psi(z)$ *be a function of the complex variable* z $(z=t+iv$, t *and* v *both real) which is regular in a circle about the point* $z=0$. *Suppose that the function* $f(t)$ *coincides with* $\psi(t)$ *for every value of the sequence* $\{t_k\}$. *Then* $f(t)$ *coincides with* $\psi(t)$ *for all real* t.

PROOF. Let the function $\psi(z)$ be regular in the circle $|z|<R$ $(R>0)$ about the point $z=0$. Then for every real t in the interval $|t|<R$ we have

$$\psi(t) = \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) \frac{t^j}{j!}$$

where α_j and β_j are real numbers. Let $A(t)$ and $B(t)$ denote respectively the real and imaginary parts of $\psi(t)$ so that

$$A(t) = \sum_{j=0}^{\infty} \alpha_j \frac{t^j}{j!} \quad \text{and} \quad B(t) = \sum_{j=0}^{\infty} \beta_j \frac{t^j}{j!} \cdot$$

We now make use of the hermitian property of $f(t)$ and the equation $f(t_k)=\psi(t_k)$ and obtain easily the relations $A(-t_k)=A(t_k)$ and $B(-t_k)=-B(t_k)$ holding for every value of the sequence $\{t_k\}$. Since the point $t=0$ is the limit point of the sequence of real numbers $\{t_k\}$, we can verify easily that

$$A(t) = \sum_{j=0}^{\infty} \alpha_{2j} \frac{t^{2j}}{(2j)!} \quad \text{and} \quad B(t) = \sum_{j=0}^{\infty} \beta_{2j+1} \frac{t^{2j+1}}{(2j+1)!}$$

for all real t in $|t|<R$.

We next introduce the functions $g(t) = f(t)f(-t)$ and $\theta(z) = \psi(z)\psi(-z)$ $(z=t+iv$, t and v real) and note the following: The function $g(t)$ is a real-valued, even and positive-definite function of the real variable t; the function $\theta(z)$ is regular in the same circle $|z|<R$ about the point $z=0$ and for real t

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$$\theta(t) = |\psi(t)|^2 = \{A(t)\}^2 + \{B(t)\}^2 = \sum_{j=0}^{\infty} \gamma_{2j} \frac{t^{2j}}{(2j)!}$$

where γ_{2j} is real and therefore $\theta(t)$ is a real-valued and even function of t. Before proceeding further we first prove the following lemma:

LEMMA. *The function $g(t)$ has finite derivatives of all orders for every real t.*

PROOF OF THE LEMMA. We can represent $g(t)$ in the form $g(t) = \int_{-\infty}^{\infty} \cos tx dG(x)$ where $G(x)$ is a nondecreasing function of bounded variation. We show first that the function $g(t)$ can be differentiated twice for any real t. We note that the relation $g(t_k) = \theta(t_k)$ holds for every value of the sequence $\{t_k\}$, $t_k \to 0$ so that we must have $g(0) = \theta(0)$ and therefore $\gamma_0 = \int_{-\infty}^{\infty} dG(x) > 0$. Thus for every value of the sequence $\{t_k\}$ $t_k \to 0$ we have

$$\frac{g(t_k) - g(0)}{t_k^2} = \frac{\theta(t_k) - \theta(0)}{t_k^2} = \frac{\gamma_2}{2} + O(t_k^2)$$

so that

$$\lim_{k\to\infty} \frac{g(t_k) - g(0)}{t_k^2} = \frac{\gamma_2}{2}$$

exists and is finite. But we note that

$$\frac{g(t_k) - g(0)}{t_k^2} = -2\int_{-\infty}^{\infty} \frac{\sin^2 \frac{t_k x}{2}}{t_k^2} dG(x)$$

and therefore

$$\int_{-\infty}^{\infty} \frac{\sin^2 \frac{t_k x}{2}}{t_k^2} dG(x)$$

tends to a finite limit as $t_k \to 0$. Then we use Fatou's Lemma and deduce immediately that $\int_{-\infty}^{\infty} x^2 dG(x) < \infty$, that is, $|g''(0)| < \infty$. But it is well known that the existence of the $2p$th derivative of a positive-definite function at the origin $t=0$ ensures that it is differentiable $2p$ times for any real t. Therefore $g''(t)$ exists and is finite for any real t. We now prove the lemma by induction. We suppose that the function $g(t)$ has finite derivatives up to an even order $2n$ and then show that the derivative of order $2n+2$ will also exist. We note that the function $g(t)-\theta(t)$ is real-valued, even and can be differentiated $2n$ times and further vanishes for every value of the sequence $\{t_k\}$, $t_k\to 0$. We now apply the Theorem of Rolle: *Between two zeros of a differentiable function there exists at least one zero of its derivative.* Therefore, it follows that the function $g'(t)-\theta'(t)$ vanishes for every value of a sequence $\{t_{k_1}\}$ of real numbers, $t_{k_1}\to 0$ which lie between the numbers of the sequence $\{t_k\}$. We thus apply Rolle's Theorem successively $2n$ times and finally conclude that the function $g^{(2n)}(t)$ coincides with the function $\theta^{(2n)}(t)$ for every value of a sequence $\{t_{k_{2n}}\}$ of real numbers, $t_{k_{2n}}\to 0$. For simplicity in notation we denote this sequence by $\{t_k'\}$. The relation $g^{(2n)}(t_k')=\theta^{(2n)}(t_k')$ holds for every value t_k', $t_k'\to 0$ so that we must have $g^{(2n)}(0)=\theta^{(2n)}(0)$ and therefore $(-1)^n\gamma_{2n}=\int_{-\infty}^{\infty}x^{2n}dG(x)>0$. Thus for every value of the sequence $\{t_k'\}$, $t_k'\to 0$ we have

$$\frac{g^{(2n)}(t_k')-g^{(2n)}(0)}{t_k'^2}=\frac{\theta^{(2n)}(t_k')-\theta^{(2n)}(0)}{t_k'^2}=\frac{\gamma_{2n+2}}{2}+O(t_k'^2)$$

so that

$$\lim_{k\to\infty}\frac{g^{(2n)}(t_k')-g^{(2n)}(0)}{t_k'^2}$$

exists and is finite. But we can verify easily that

$$\frac{g^{(2n)}(t_k')-g^{(2n)}(0)}{t_k'^2}=(-1)^{n+1}2\int_{-\infty}^{\infty}x^{2n}\frac{\sin^2\dfrac{t_k'x}{2}}{t_k'^2}dG(x)$$

so that the integral on the right-hand side tends to a finite limit as $t_k'\to 0$. We apply again Fatou's Lemma and deduce that $\int_{-\infty}^{\infty}x^{2n+2}dG(x)<\infty$ that is, $g^{(2n+2)}(0)$ exists. Therefore $g^{(2n+2)}(t)$ exists and is finite for any real t, thus completing the induction.

Now we turn to the proof of the theorem. We note that the function $g(t)$ has finite derivatives of all orders and further $g^{(2n)}(0)=\theta^{(2n)}(0)$ for every n. Hence we have

$$\limsup_{n\to\infty}\left[\frac{|g^{(2n)}(0)|}{(2n)!}\right]^{1/2n}=\limsup_{n\to\infty}\left[\frac{|\theta^{(2n)}(0)|}{(2n)!}\right]^{1/2n}=\frac{1}{R}$$

so that the positive definite function $g(z)$ as a function of the complex variable z is also regular in the circle $|z|<R$ about the point $z=0$. Then it follows immediately from the theorem of Raikov [3] that the positive definite function $f(z)$ is also regular in the same circle

$|z| < R$ about the point $z = 0$. Thus we conclude that both the functions $f(z)$ and $\psi(z)$ are regular in the circle $|z| < R$ and further note that they coincide for every value of a sequence $\{t_k\}$ of real numbers, $t_k \to 0$. Therefore the functions $f(z)$ and $\psi(z)$ coincide for every z in the circle $|z| < R$. It then follows at once from the theorem due to Boas [1] that the function $f(t)$ coincides with $\psi(t)$ for every real t and further the function $f(z)$ (z complex) is regular in the strip $|\operatorname{Im} z| < R$.

The following corollary is an immediate consequence of the above theorem:

COROLLARY. *Let $f(t)$ be a real-valued, even and positive-definite function of the real variable t and let $\psi(z)$ be a regular function of the complex variable z such that $\psi(t)$ is real-valued and even. Suppose that $f(t)$ coincides with $\psi(t)$ for every value of the sequence $\{t_k\}$ of real numbers, $t_k \to 0$. Then $f(t)$ coincides with $\psi(t)$ for every real t.*

This result has been proved by Linnik in [2] under the additional condition that the function $\psi(t)$ is also a positive-definite function.

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CATHOLIC UNIVERSITY OF AMERICA