EXAMPLES OF p-ADIC TRANSFORMATION GROUPS

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1. **Introduction.** Our purpose here is to outline the construction of an n-dimensional space X^n, $n \geqq 2$, upon which the p-adic group A_p acts so that the orbit space X^n/A_p is of dimension $n+2$. Though such examples are new, it had been known [**1**; **4**], that either they do exist or a certain long standing conjecture on transformation groups must be true. The conjecture states that every compact effective group acting on a (generalized) manifold must be a Lie group; it may well be false.

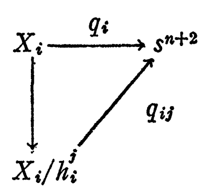
Another question concerns the amount, k, by which the (cohomology) dimension of a compact space can be raised under the decomposition map $X \to X/A_p$. By [**1**; **4**], $k \leqq 3$. (An example in which $k=1$ is essentially due to Kolmogoroff [**2**].) No example is known for which $k=3$. The authors expect to have more to say on this subject.

2. **The building blocks.** There exists an $(n+2)$-dimensional complex X_i, a homeomorphism $h_i: X_i \to X_i$, maps $q_i: X_i \to s^{n+2}$ and $r_i: X_i \to X_i$ such that (S^{n+2} is the standard $n+2$-simplex)

(a) h_i is of period p^i.

(b) $q_i^{-1}(j \text{ skeleton}) = X_i(j)$ is a j-complex, $j = 0, 1, \cdots, n+2$.

(c) $q_i h_i = q_i$ so that q_i can be factored through the orbit space: $j = 1, p, \cdots, p^{i-1}$



(d) q_{ij*}: $H_{n+2}(X_i/h_i^j,\ X_i(n+1)/h_i^j;\ Z_p) \to H_{n+2}(s^{n+2},\ \dot{s}^{n+2};\ Z_p)$ is onto; and

(e) $r_i: X_i \to X_i(n)$ is a retraction.

3. (E, π, B, X, q). Let m be a positive integer and s_0 be the standard m-simplex. We consider 5-tuples (E, π, B, X, q), where E, B, and X are m-complexes and π, q are simplical maps such that

A1. $\pi: E \to B$, $q: X \to s_0$,

A2. for each m-simplex $s \in B$, there are specified maps

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$$\psi_s\colon X \to \pi^{-1}(s) = E_s,$$
$$\phi_s\colon s_0 \to s$$

where ϕ_s is a linear homeomorphism;

A3. commutativity holds in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi_s} & E_s \\ q\downarrow & & \downarrow \pi \mid E_s. \\ s_0 & \xrightarrow[\phi_s]{} & s \end{array}$$

A4. For s, s' two m-simplexes of K,

$$\psi_s \mid X_{s\cap s'} = \psi_{s'} \mid X_{s\cap s'}, \quad \text{and} \quad \phi_s^{-1} \mid s \cap s' = \phi_{s'}^{-1} \mid s \cap s',$$

where $X_{s\cap s'} = q^{-1}\phi_s^{-1}(s\cap s')$.

We need two results concerning such 5-tuples.

3.1. *Given X, q, and a complex B which is the barycentric subdivision of a complex, there exists a complex E and a map $\pi\colon E\to B$ such that (E, π, B, q, X) satisfies* A1–A4 *and such that the ψ_s's are homeomorphisms.*

3.2. *If in addition to* A1–A4, *we assume*

A5. $H_m(X,\ q^{-1}(\dot{s});\ G) \to^{q_*} H_m(s,\ \dot{s};\ G)$ *is onto, then* $H_m(E;\ G) \to H_m(B;\ G)$ *is onto.*

4. **The example.** X^n is defined as the inverse limit of a sequence

$$E_0 \xleftarrow[\pi_1]{} E_1 \xleftarrow[\pi_2]{} E_2 \leftarrow \cdots$$

such that on each E_i (an $(n+2)$-complex) we have a map $\bar{h}_i$ of period p^i. The E_i are defined inductively. E_0 is taken to be a triangulated $(n+2)$-sphere, and $\bar{h}_0 =$ identity. Suppose E_i, π_i, $\bar{h}_i$ have been defined. Then set $B_{i+1} =$ the barycentric subdivision of E_i and use 3.1 relative to B_{i+1}, X_{i+1}, q_{i+1} (see §2), to obtain E_{i+1}, π_{i+1}. The homeomorphism $\bar{h}_{i+1}$ is defined on E_{i+1} in terms of h_{i+1}, π_{i+1} and $\bar{h}_i$, so that we have the equivariance

$$\pi_{i+1}\bar{h}_{i+1} = \bar{h}_i\pi_{i+1}.$$

Thus the map

$$(e_0, e_1, \cdots) \to (\bar{h}_0 e_0, \bar{h}_1 e_1, \cdots)$$

defines an effective action of the p-adic group on X^n.

It is next shown that $\overline{E}_i = E_i/\bar{h}_i$, $i = 1, 2, \cdots$ satisfy axioms

A1–A5 of §3, where $\bar{\pi}_i\colon E_i/\bar{h}_i \to E_{i-1}/\bar{h}_{i-1}$ is defined via (2c), and the ψ_s's are essentially like the q_{ij}'s in (2c). By (3.2) all homeomorphisms in the sequence

$$H_{n+2}(\overline{E}_0; Zp) \underset{\bar{\pi}_{1*}}{\longleftarrow} H_{n+2}(\overline{E}_1; Zp) \leftarrow \cdots$$

are onto so that the orbit space, X/A_p, which is the inverse limit of the sequence

$$E_0/\bar{h}_0 \underset{\bar{\pi}_1}{\leftarrow} E_1/\bar{h}_0 \underset{\bar{\pi}_2}{\leftarrow} E_2/\bar{h}_2 \underset{\bar{\pi}_3}{\leftarrow} \cdots,$$

is $(n+2)$-dimensional.

Finally, it follows from (2e) that X^n is n-dimensional.

Bibliography

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