RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

POLYHEDRAL HOMOTOPY-SPHERES

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It has been conjectured that a manifold which is a homotopy sphere is topologically a sphere. This conjecture has implications, for example, in the theory of differentiable structures on spheres (see, e.g., [3, p. 33]).

Here I shall sketch a proof of the following theorem:

Let M be a piecewise-linear manifold of dimension $n \geqq 7$, which has the same homotopy-type as the n-sphere S^n. Then there is a piecewise-linear equivalence of $M - \{point\}$ with euclidean n-space; in particular, M is topologically equivalent to S^n.

This theorem is not the best possible, for C. Zeeman has been able to refine the method presented here so as to prove the same theorem for $n \geqq 5$.

A *piecewise-linear n-manifold* is a polyhedron with a linear triangulation satisfying the condition that the link of each vertex is combinatorially equivalent to the standard $(n-1)$-sphere; all the manifolds with which I am concerned here have no boundary. In general, all the spaces in this paper will be polyhedra, finite or infinite, and each map will be polyhedral, i.e., induced by a simplicial map of linear triangulations.

Let K be a finite subpolyhedron of the finite polyhedron L; let K' be a finite subpolyhedron of the finite polyhedron L'; let $f: L \to L'$ be a polyhedral map. f is called a *relative equivalence* $(L, K) \Rightarrow (L', K')$, if $f(K) \subset K'$ and $L-K$ is mapped by f in a 1-1 manner onto $L'-K'$.

Recall J. H. C. Whitehead's definition of contraction [7, p. 247]: If the simplicial complex A has a simplex σ^p which is the face of just one simplex τ^{p+1}, and B is the simplicial complex obtained from A by removing the open simplexes σ^p and τ^{p+1}, then $A \to B$ is called an *elementary contraction* at σ^p. A finite sequence of elementary contractions is a *contraction.*

If K is a finite subpolyhedron of the finite polyhedron L, then it is said that L *contracts onto* K, if there is a linear triangulation A of L,

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such that a subcomplex B of A triangulates K, and such that $A \to B$ is a contraction.

LEMMA 1. *If L contracts onto K and $(L, K) \Rightarrow (L', K')$ is a relative equivalence, then L' contracts onto K'.*

This can be shown by the methods of Whitehead [cf. **6**, Theorem 1; **7**, Theorem 6 and Theorem 7].

LEMMA 2. *Let M be a piecewise-linear manifold, and let L be a subpolyhedron which contracts onto $K \subset L$, and let U be a neighborhood of K in M. Then there is a piecewise-linear equivalence $h: M \to M$, such that $h(U)$ is a neighborhood of L* [cf. **7**, Theorem 23].

The proof can be reduced to the case where $B \subset A \subset C$ are triangulations of $K \subset L \subset M$ respectively, and $A \to B$ is an elementary contraction at σ^p. It can then be reduced, due to the local euclidean nature of M and the local nature of an elementary contraction, to the case when M is euclidean space; the proof in this case is obvious.

An *n-element* E is a polyhedron equivalent to the standard geometric n-simplex. Int E will denote the subset of E which corresponds to the interior of the n-simplex.

The following lemma was noticed, for $n=3$, by Moise [**4**].

LEMMA 3. *Let the polyhedron M contain two n-elements E_1 and E_2, such that $M = \text{Int } E_1 \cup \text{Int } E_2$. Then $M - \{\text{point}\}$ is piecewise-linearly equivalent to euclidean n-space; in particular M is topologically a sphere.*

This can be proved by the methods of B. Mazur [**2**] or M. Brown [**1**].

If K is a finite polyhedron and Q is the nonsingular join of K to a point x, then Q is called the *cone* on K with vertex x. If L is a finite subpolyhedron of K, then the cone Q_1 on L with vertex x is called a *subcone* of Q.

LEMMA 4. *A cone Q contracts onto any subcone Q_1.*

LEMMA 5. *If P is a finite subpolyhedron of the cone Q, and* $\dim P \leqq p$, *then there is a subcone Q_1 of Q, such that $P \subset Q_1$ and* $\dim Q_1 \leqq p+1$.

"dim" denotes dimension. The proofs of Lemmas 4 and 5 are elementary.

Let K be a finite polyhedron, M an n-manifold, and $f: K \to M$ a polyhedral map. Then K is the union of a finite number of convex sets $\{\gamma_i\}$, on each of which f is linear. f is said to be in *general position* if there exists such $\{\gamma_i\}$ that for all i, j,

(1) If $\dim \gamma_i + \dim \gamma_j < n + \dim \gamma_i \cap \gamma_j$, then $f | \gamma_i \cup \gamma_j$ is 1-1;

(2) If $\dim \gamma_i + \dim \gamma_j \geqq n + \dim \gamma_i \cap \gamma_j$, then $\dim (\gamma_i \cap f^{-1} f \gamma_j) \leqq \dim \gamma_i + \dim \gamma_j - n$.

The *singular set* of $f: K \rightarrow M$, $S(f)$, is the closure in K of the set $\{x \in K | f^{-1}fx$ contains more than one point$\}$. The following lemma follows from property 2 of general position.

LEMMA 6. *If K is k-dimensional, and $f: K \rightarrow M$ is in general position, where M is an n-dimensional manifold, then* $\dim S(f) \leqq 2k - n$.

LEMMA 7. *If K is a subpolyhedron of L, and M is a manifold, and $f: L \rightarrow M$ is a map such that $f | K$ is in general position, then there is a map in general position $g: L \rightarrow M$ such that $g | K = f | K$.*

The proof is obtained by localizing to the well-known proof for the case that M is a euclidean space (cf. [5]).

LEMMA 8 (PENROSE-WHITEHEAD-ZEEMAN [5]). *Let A be a subpolyhedron of the manifold M, with $2(\dim A + 1) \leqq \dim M = n$, and let A be contractible to a point in M. Then A is contained in the interior of an n-element in M.*

The proof consists in embedding the cone on A, Q, in M. There exists a map $f: Q \rightarrow M$, such that $f | A =$ inclusion; by Lemma 7, assume f is in general position. By Lemma 6, f will be nonsingular except in the case $2(\dim A + 1) = \dim M$, when there will be 0-dimensional singularities, which can be removed by a trick. Hence $Q \subset M$; Q contracts to a point; a point in M is contained in the interior of an n-element. By Lemma 2, Q (and hence A) is contained in the interior of an n-element.

Let T be a linear triangulation of an n-manifold M; T_p will denote the *p-skeleton*. T^* will denote the *dual cell complex*; T_q^* its q-skeleton.

LEMMA 9. *Let T be a linear triangulation of the n-manifold M; let U and V be neighborhoods of T_p and T_q^* respectively, where $p + q \geqq n - 1$. Then there is a polyhedral equivalence $g: M \rightarrow M$, such that $M = U \cup gV$.*

This is proved by embedding M nicely in the join of T_p and T_q^* and applying a similar, elementary, lemma to that join.

PROOF OF THEOREM.

(a) CASE. $n = 2k + 1$, $n \geqq 7$.

Let T be a linear triangulation of M. Let Q be the cone on T_k. Let $f: Q \rightarrow M$ be a map in general position such that $f | T_k$ is the inclusion of T_k into M; such a map exists by Lemma 7 and the fact that M is k-connected.

By Lemma 6, since dim $Q=k+1$, and dim $M=2k+1$, it follows that dim $S(f)\leqq 1$. By Lemma 5, there is a subcone $Q_1\subset Q$, such that $S(f)\subset Q_1$ and dim $Q_1\leqq 2$.

By the theorem of Penrose, Whitehead, and Zeeman (Lemma 8), since dim $fQ_1\leqq 2$ and dim $M\geqq 6$, there is an n-element $E\subset M$ containing fQ_1 in its interior.

Q contracts into Q_1 (Lemma 4); since $S(f)\subset Q_1$, f defines a relative equivalence $(Q, Q_1)\Rightarrow(fQ, fQ_1)$; by Lemma 1, fQ contracts onto fQ_1. Since Int E is a neighborhood of fQ_1, by Lemma 2, there is a piecewise-linear homeomorphism $h\colon M\rightarrow M$ such that $fQ\subset h(\text{Int } E)$.

Hence $\Delta_0=hE$ is an n-element which is a neighborhood of fQ. $T_k\subset fQ$; hence Δ_0 is a neighborhood of T_k.

Similarly, an n-element Δ_0^* may be found, which is a neighborhood of T_k^*.

By Lemma 9, there is a piecewise-linear homeomorphism $g\colon M\rightarrow M$ such that $M=\text{Int }\Delta_0\cup\text{Int } g\Delta_0^*$. Let $\Delta_1=g\Delta_0^*$. M is the union of the interiors of the two n-elements Δ_0 and Δ_1; hence by the Mazur-Brown Theorem (Lemma 3), the complement of a point of M is polyhedrally equivalent to euclidean n-space. In particular M is topologically a sphere.

(b) CASE. $n=2k$, $n\geqq 8$.

The proof is very similar; the same notation is used. In this case, however, dim $S(f)\leqq 2$; dim $Q_1\leqq 3$. The Penrose-Whitehead-Zeeman Theorem applies to fQ_1 since dim $fQ_1\leqq 3$, and dim $M\geqq 8$. The rest of the proof is word for word the same.

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