COMBINATORIAL TOPOLOGY OF AN ANALYTIC FUNCTION ON THE BOUNDARY OF A DISK

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Preliminaries. A complex valued function $\zeta(t)$ defined on an oriented circle S of circumference c, t the usual distance parameter, $0 \leqq t < c$, is a regular representation if it possesses a continuous non-vanishing derivative $\zeta'(t)$. An image point ζ_0 is a *simple crossing point* if there exist exactly two distinct numbers t_0' and t_0'' such that $\zeta(t_0') = \zeta(t_0'') = \zeta_0$ and if the tangents $\zeta'(t_0')$ and $\zeta'(t_0'')$ are linearly independent. A regular representation is *normal* (Whitney) if it has a finite number of simple crossing points and has for every other image point ζ but one preimage point t. A pair of representations $\tilde{\zeta}$ and ζ are *topologically equivalent* if there exists a sense-preserving homeomorphism h of S onto S such that $\tilde{\zeta} = \zeta \circ h$.

A mapping F of a disk D, $|z| < R$, is *open* if, for every open set U in D, $F(U)$ is open in the plane; F is *light* if the preimage of each image point is totally disconnected; F is *properly interior* on $\overline{D}$, $|z| \leqq R$, if F is continuous on $\overline{D}$, $F|$ bdy D is locally topological, F is sense-preserving, light and open on D. It can be shown (using results of Carathéodory, Stoilow, Whyburn) that *given a properly interior mapping F there exists an analytic function W on D that is locally topological near and on* bdy D *and there exists a sense-preserving homeomorphism H of $\overline{D}$ onto $\overline{D}$ such that $F = W \circ H$.*

A representation ζ will be called an *interior boundary* [*analytic boundary*] if ζ is locally topological and if there exists a properly interior mapping F [an analytic function W that is locally topological near and on bdy D] such that $F(Re^{it}) \equiv \zeta(t)$ [$W(Re^{it}) \equiv \zeta(t)$]. Thus, *every interior boundary is topologically equivalent to an analytic boundary.*

The problem probably first arose in the study of the Schwartz-Christoffel mapping function (Schwartz, Schlaefli, Picard) and, in this context, was formulated essentially as follows.

Let $Z_0, Z_1, \cdots, Z_{n-1}$ be a sequence of n-distinct complex numbers which are in general position. By connecting these points consecutively from Z_k to Z_{k+1}, mod n, a closed oriented polygon is formed. Let $\alpha_k\pi$ be the angle from $Z_k - Z_{k-1}$ to $Z_{k+1} - Z_k$ with $-1 < \alpha_k < 1$. Then for any set of n real number and any complex number $A \neq 0$ the function

$$\Phi(Z) = A(Z - a_0)^{-\alpha_0}(Z - a_1)^{-\alpha_1} \cdots (Z - a_{n-1})^{-\alpha_{n-1}},$$

with $-\pi/2 < \arg(Z-\alpha_k) < \pi/2$, is an analytic function on the upper half plane; furthermore

$$W(z) = \int_i^z \Phi(Z)dZ + B$$

is also analytic there and maps the real axis onto a possibly different polygon with $W(a_k) = Z_k'$ but with $Z_k' - Z_{k-1}'$ having the same direction as $Z_k - Z_{k-1}$.

PROBLEM A (EMILE PICARD, *Traité d'analyse*, vol. 2, p. 313). *Find necessary and sufficient conditions on $Z_0, Z_1, \cdots, Z_{n-1}$ so that there exist complex numbers A, B and real numbers $a_0, a_1, \cdots, a_{n-1}$ so that $W(a_k) = Z_k$, and thus that the real axis is mapped onto the polygon determined by the given Z_k.* (Actually Schlaefli and others were concerned also with the problem of finding an effective method for determining the a_k.)

Some time ago a clearly related problem was formulated by Loewner (circa 1948) which will be stated in the form:

PROBLEM B (CHARLES LOEWNER). *Given a normal representation ζ of a closed curve find necessary and sufficient conditions that ζ be equivalent to an analytic boundary (or, what is the same thing, that ζ be an interior boundary).*

Problem A is a corollary of Problem B. In this paper a solution to Problem B is announced. More precisely Problem A was concerned only with oriented polygons with a tangent winding number of one.

STATEMENT OF RESULTS. *In the following ζ will always be a normal representation*; the simple crossing points will be called *vertices*. Let $\tau[\zeta]$ be the tangent winding number of ζ and let $\omega(\zeta, \pi)$ be the winding number (index) of ζ about a point π, $\pi \notin [\zeta]$. The *outer boundary* of ζ is the subset of $[\zeta]$ which is contained in the closure of the unbounded component of the complement of $[\zeta]$; a point π is a *positive outer point* if π is on the outer boundary and is not a vertex and if there exist points π' arbitrarily close to π such that $\omega(\zeta, \pi') = +1$.

LEMMA. *If ζ is an interior boundary then $\tau[\zeta] \geqq 1$ and $\omega(\zeta, \pi) \geqq 0$ for all $\pi \in [\zeta]$.*

Because of this Lemma only curves ζ which satisfy these conditions need be considered; *call this class C^+.*

Begin at a positive outer point $\pi = \zeta(0)$ and traverse the curve in the direction of its sense. Index the vertices using consecutively the integers from 0 to $n-1$, $\zeta_0, \zeta_1, \cdots, \zeta_{n-1}$. Let the $2n$ preimages of the vertices be denoted by s_k and index so that $0 < s_0 < s_1 < \cdots < s_{2n-1} < c$. If $\zeta(s_j) = \zeta(s_k)$, $s_j \neq s_k$, s_j is also denoted by s_k^* (and s_k by s_j^*). Let ν_k be defined, with $\zeta(t) = \xi(t) + i\eta(t)$, by

$$\nu_k = \nu(s_k) = \operatorname{sgn} \begin{vmatrix} \xi'(s_k^*) & \eta'(s_k^*) \\ \xi'(s_k) & \eta'(s_k) \end{vmatrix}.$$

If the sequence $\{s_k\}$ together with the * operation, the ν_k and the fact that $\zeta(0) = \pi$ is a positive outer point are given then the oriented curve represented by ζ is determined up to a sense preserving homeomorphism of the plane onto itself (follows from e.g. Adkisson and MacLane and Gehman). See Figure 1 in which

$$\nu_0 = \nu_1 = \nu_3 = \nu_5 = 1, \qquad \nu_2 = \nu_4 = \nu_6 = \nu_7 = -1;$$

$$s_0^* = s_7, \qquad s_1^* = s_6, \qquad s_2^* = s_3, \qquad s_4^* = s_5.$$

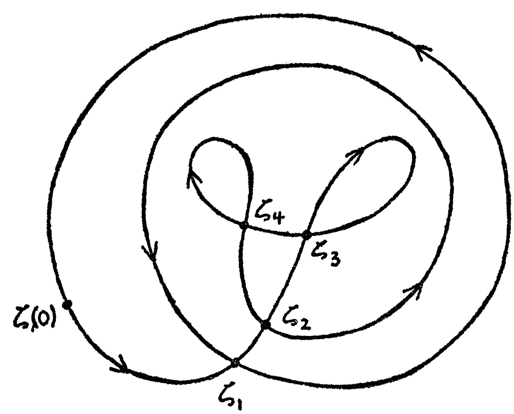

FIG. 1

Select any point π on the outer boundary of ζ, $\zeta(0) = \pi$, $\zeta \in C^+$ and let s_k be the number with the smallest index so that $\nu_k = -1$. At least one of the following situations must arise:

CASE I. $s_k^* < s_k$.

CASE II. $s_k^* > s_k$.

In the later case for each choice of an $s_j < s_k$ there corresponds one of the two situations II′ and II″:

CASE II′. $s_k^* > s_k$ and $s_j < s_k < s_k^* < s_j^*$,

CASE II″. $s_k^* > s_k$ and $s_j < s_k < s_j^* < s_k^*$.

In each of these situations a cut is defined that breaks ζ up into a pair of piecewise regular representations ζ^* and ζ^{**}. (It turns out that ζ^* and ζ^{**} can be smoothed and altered slightly so that each becomes a normal representation. This step, which is bothersome and simple technically, will be omitted here; in what follows the reader may ignore this problem and pretend that ζ^* and ζ^{**} have already been made normal.)

In Case I define on circles of circumference c^* and c^{**}:

$$\zeta^*(t) = \zeta(t + s_k^*), \qquad 0 \leqq t \leqq s_k - s_k^* = c^*;$$

$$\zeta^{**}(t) = \begin{cases} \zeta(t), & 0 \leqq t \leqq s_k^*, \\ \zeta(s_k - s_k^* + t), & s_k^* \leqq t \leqq c - s_k + s_k^* = c^{**}. \end{cases}$$

In Case II select $s_j < s_k$ and if $s_j < s_k < s_k^* < s_j^*$ we have Case II′; define on circles of circumference c^* and c^{**}:

$$\zeta^*(t) = \begin{cases} \zeta(t + s_j), & 0 \leqq t \leqq s_k - s_j, \\ \zeta(s_j + s_k - s_k^* + t), & s_k - s_j \leqq t \leqq s_k - s_j + s_j^* - s_k^* = c^*; \end{cases}$$

$$\zeta^{**}(t) = \begin{cases} \zeta(t), & 0 \leqq t \leqq s_k^*, \\ \zeta(s_k - s_k^* - t), & s_k^* \leqq t \leqq s_k^* + s_k - s_j, \\ \zeta(s_j + s_j^* - s_k - s_k^* + t), & \\ \qquad s_k^* + s_k - s_j \leqq t \leqq c + s_j + s_j^* - s_k - s_k^* = c^{**}; \end{cases}$$

but if in Case II, $s_j < s_k < s_j^* < s_k^*$, one has Case II″ and define

$$\zeta^*(t) = \begin{cases} \zeta(s_j^* + t), & 0 \leqq t \leqq s_k^* - s_j^*, \\ \zeta(s_j^* + s_k^* + s_k - t), & s_k^* - s_j^* \leqq t \leqq s_k + s_j + s_k^* - s_j^* = c^*; \end{cases}$$

$$\zeta^{**}(t) = \begin{cases} \zeta(t), & 0 \leqq t \leqq s_j^*, \\ \zeta(s_j - s_j^* + t), & s_j^* \leqq t \leqq s_j^* - s_j + s_k, \\ \zeta(s_j - s_j^* + s_k^* - s_k + t), & \\ \qquad s_j^* - s_j + s_k \leqq t \leqq c + s_j^* - s_j + s_k - s_k^* = c^{**}. \end{cases}$$

These three cuts are illustrated in Figures 2, 3 and 4.

Assuming the ζ^* and ζ^{**} altered so that they are normal (as commented upon parenthetically above) the cut process can be continued so long as the new representations remain in C^+. A normal representation ζ *possesses a complete cut sequence* provided that the representations generated by successive cuts always remain in C^+; thus ultimately the representations (in the slightly altered form) describe simple closed positively oriented curves.

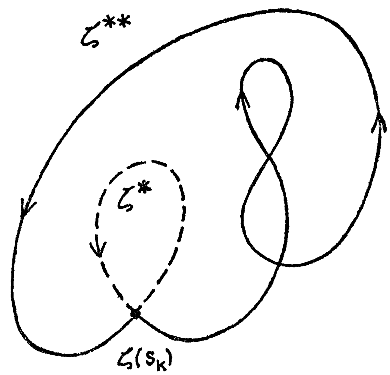

FIG. 2

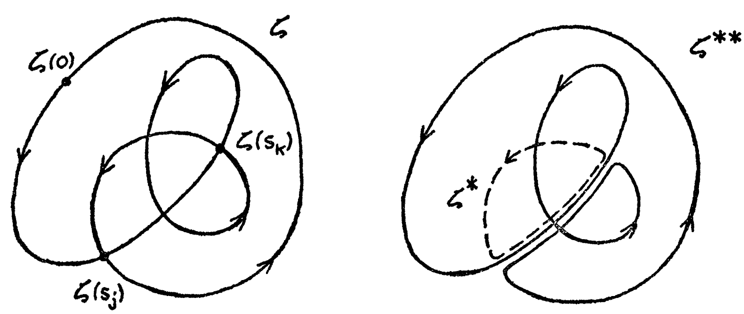

FIG. 3

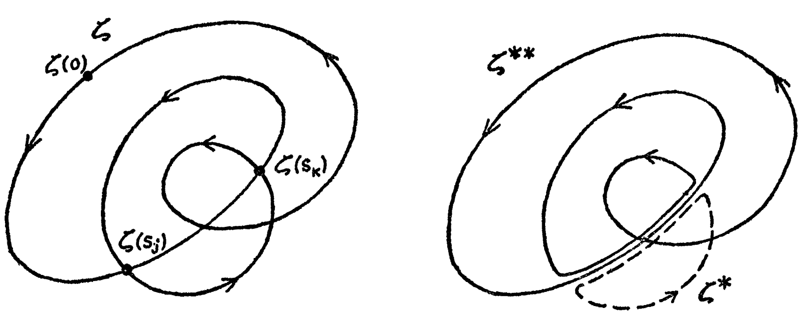

FIG. 4

PRINCIPAL LEMMA. *A normal representation* ζ *is an interior boundary if and only if* (i) *there exists a cut of type* I *and the corresponding* ζ^* *and* ζ^{**} *are both interior boundaries, or* (ii) *there does not exist a cut of type* I (*whence there must exist cuts of type* II) *but there exists an* s_j *and a corresponding cut of type* II′ *or* II″ *so that* ζ^* *and* ζ^{**} *are interior boundaries.*

It is also true that the slightly altered ζ^* and ζ^{**} have strictly less vertices than the original ζ.

It follows directly from this Lemma that

THEOREM 1. *A normal representation ζ is an interior boundary if and only if ζ possesses a complete cut sequence.*

Let μ be the number of cuts of type I required in a complete cut sequence for a given interior boundary ζ.

THEOREM 2. *If W is an analytic function which extends a representation equivalent to ζ to the disk then $W(z)$ has precisely μ zeros (counting multiplicity) in the disk, (thus e.g., $\tau[\zeta]=\mu+1$).*

COROLLARY. *ζ has a complete cut sequence with $\mu=0$ (no cuts of type I) if and only if there is a sense-preserving local homeomorphism F which extends ζ to the disk.*

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