A FAMILY OF SIMPLE GROUPS ASSOCIATED WITH THE SIMPLE LIE ALGEBRA OF TYPE (G_2)[1]

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The purpose of this note is to announce the construction of a family of simple groups which seem to be new. The family contains finite as well as infinite groups. The orders of the finite groups in the family are $q^3(q-1)(q^3+1)$, where $q=3^{2n+1}$, $n=1, 2, 3, \cdots$. The construction is carried out by applying to the Chevalley groups of type (G_2) a method which emerges naturally when one looks at Suzuki's [3] construction of his simple groups from the Lie-theoretical point of view.

For the sake of the reader who is not familiar with the Lie theory, we shall define, in the last paragraph of this paper, the groups purely in terms of matrices.

Let $\mathfrak{g}$ be the simple Lie algebra of type (G_2) over the complex number field, and $\Sigma=\{\pm a, \pm b, \pm(a+b), \pm(2a+b), \pm(3a+b), \pm(3a+2b)\}$ the set of roots of $\mathfrak{g}$. Let $\mathfrak{g}_K$ be the corresponding algebra over an arbitrary field K, and define automorphisms $x_r(t)$, where $r\in\Sigma$, $t\in K$, of $\mathfrak{g}_K$ as in [1]. Denote by G the group generated by the $x_r(t)$, $r\in\Sigma$, $t\in K$. Let P be the additive group generated by Σ, and K^* the multiplicative group of nonzero elements in K. As in [1], associate with each homomorphism $\chi: P\to K^*$ an automorphism $h(\chi)$ of $\mathfrak{g}_K$. It is known [1] that G contains all the $h(\chi)$. Also, G contains an element ω_0 such that $\omega_0^2=1$, $\omega_0 x_r(t)\omega_0^{-1}=x_{-r}(-t)$ for all $r\in\Sigma$, $t\in K$.

Now let K be a field of characteristic 3 which has an automorphism $t\to t^\theta$ such that $3\theta^2=1$, the identity automorphism. Note that any finite field of $q=3^{2n+1}$, $(n\geqq 1)$, elements has the above property with $t^\theta=t^m$, where $m=3^n$. Now define the group G^1 to be the subgroup of G generated by ω_0 and the elements of the form

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})x_{2a+b}(t^{2\theta+1}), \qquad t\in K.$$

Then we have:

(1) *If K has more than three elements, then G^1 is simple.*

(2) *The subgroup $\mathfrak{U}^1$ generated by the $\alpha(t)$, $t\in K$, contains all elements of the forms*

[1] This work was done while the author held a Research Associateship of the Office of Naval Research, U. S. Navy.

$$\beta(t) = x_{a+b}(t^\theta)x_{3a+b}(t);\ \gamma(t) = x_{2a+b}(t^\theta)x_{3a+2b}(t).$$

Every element in $\mathfrak{U}^1$ *is written uniquely as* $\alpha(t)\beta(s)\gamma(u)$, t, s, $u \in K$.

(3) *If K is finite, then the group* $\mathfrak{H}^1$ *consisting of all elements of the form* $h(\chi)$, *where* $\chi(a)=\chi(b)^\theta$, *is contained in* G^1. *Every element in the group generated by* $\mathfrak{U}^1$ *and* $\mathfrak{H}^1$ *is written uniquely as* uh, *where* $u\in\mathfrak{U}^1$, $h\in\mathfrak{H}^1$.

(4) *Every element of* G^1 *not in the group* $\mathfrak{U}^1\mathfrak{H}^1$ *is written uniquely as* $uh\omega_0 v$, *where* $u, v\in\mathfrak{U}^1$, $h\in\mathfrak{H}^1$.

The order of G^1 for finite K can be computed easily from (2), (3) and (4). The proof of (1)–(4) is based on the following proposition.

(5) *The group G admits an automorphism* $x\to\bar{x}$ *of order 2 such that*

$$(x_a(t))^- = x_b(t^{3\theta}), \qquad (x_{-a}(t))^- = x_{-b}(t^{3\theta}),$$
$$(x_{a+b}(t))^- = x_{3a+b}(t^{3\theta}), \qquad (x_{-(a+b)}(t))^- = x_{-(3a+b)}(t^{3\theta}),$$
$$(x_{2a+b}(t))^- = x_{3a+2b}(t^{3\theta}), \qquad (x_{-(2a+b)}(t))^- = x_{-(3a+2b)}(t^{3\theta}),$$
$$(h(\chi))^- = h(\bar{\chi}),$$

where $\bar{\chi}$ *is defined by* $\bar{\chi}(a)=\chi(b)^\theta$, $\bar{\chi}(b)=\chi(a)^{3\theta}$.

It turns out that G^1 is the set of elements x in G such that $\bar{x}=x$, provided that K is finite.

Since the group G has a faithful matric representation (cf. [2]), the group G^1 can be described purely in terms of matrices. Let $E_{\mu\nu}$, $\mu,\nu=0, \pm1, \pm2, \pm3$, be the 7×7 matrices with elements in K such that $E_{\mu\nu}E_{\kappa\lambda}=\delta_{\nu\kappa}E_{\mu\lambda}$, where $\delta_{\nu\kappa}$ is the Kronecker delta. Set

$$X_{ij} = E_{-i,-j} - E_{ji}, \qquad (i\neq j),$$
$$X_{i0} = 2E_{-i0} - E_{0i} + E_{i',-i''} - E_{i'',-i'},$$
$$X_{0i} = -2E_{i0} + E_{0,-i} - E_{-i',i''} + E_{-i'',i'},$$

where $i, j=1, 2, 3$, and where (i, i', i'') is an even permutation of $(1, 2, 3)$. For $t\in K$, set

$$x_{ij}(t) = I + tX_{ij},$$
$$x_{i0}(t) = I + tX_{i0} - t^2E_{-ii},$$
$$x_{0i}(t) = I + tX_{0i} - t^2E_{i,-i},$$

where I is the identity matrix. Then the group G^1 can be defined as the subgroup of $GL(7, K)$ generated by

$$\Omega_0 = x_{10}(1)x_{23}(1)x_{01}(-1)x_{32}(-1)x_{10}(1)x_{23}(1)$$

and the matrices of the form

$$A(t) = x_{20}(t^\theta)x_{12}(t)x_{10}(t^{\theta+1})x_{03}(t^{2\theta+1}).$$

References

1. C. Chevalley, *Sur certains groupes simples*, Tôhuku Math. J. (2) vol. 7 (1955) pp. 14–66.

2. Rimhak Ree, *On some simple groups defined by C. Chevalley*, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 392–400.

3. Michio Suzuki, *A new type of simple groups of finite order*, Proc. Nat. Acad. Sci. U.S.A. vol. 46 (1960) pp. 868–870.

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RESEARCH PROBLEM

1. Sherman K. Stein: *Partitions.*

Let $p(n)$ be the number of partitions of n and let m be an integer, $1 \leqq m \leqq p(n)$. Is there a subset $A(m, n)$ of the integers, such that the number of partitions of n into elements from $A(m, n)$ is precisely m?
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