1. Let \( C \) be the complex plane, \( S(C) \) the tribe of all Borel parts of \( C \), \( B^\infty(C) \) the algebra of bounded complex-valued Borel measurable functions defined on \( C \) and \( M^1(C) \) the set of bounded complex Radon measures on \( C \). Let \( E \) be a locally convex space which is separated, quasi-complete and barrelled. A family \( \mathcal{F} = (\mu_x(x'))_{x \in E, x' \in E'} \) of measures belonging to \( M^1(C) \) is called a \textit{spectral family} on \( C \) if there exists a representation \( f \mapsto U_{\mathcal{F}, f} \) of the algebra \( B^\infty(C) \) into the algebra \( L(E, E) \) mapping \( 1 \) onto \( I \) and satisfying the equations \( \int_{C} \phi \, d\mu_x(x') = \langle U_{\mathcal{F}, f} \phi, x' \rangle \) for all \( \phi \in B^\infty(C) \), \( x \in E \), \( x' \in E' \). By \( P_{\mathcal{F}} \) we denote the \textit{spectral measure} defined on \( S(C) \) by the equations \( P_{\mathcal{F}}(\sigma) = U_{\mathcal{F}, \phi_\sigma} \) (\( \phi_\sigma \) is the characteristic function of \( \sigma \)). A linear mapping \( T \) of (the vector space) \( D_T \subset E \) into \( E \) commutes with \( \mathcal{F} \) if \( TU_{\mathcal{F}, f} = U_{\mathcal{F}, f} T \) for all \( f \in B^\infty(C) \).

Let \( T \) be a linear mapping of \( D_T \subset E \) into \( E \). We say that \( \lambda \in \hat{C} \) (= the one point compactification of \( C \)) belongs to the \textit{resolvent set} \( r(T) \) of \( T \) if there is a neighborhood \( V \) of \( \lambda \) such that: (i) \( \lambda I - T \) is a one-to-one mapping of \( D_T \) onto \( E \) and \( (\lambda I - T)^{-1} \in L(E, E) \) for each \( \lambda \in V - \{\infty\} \); (ii) \( \{ (\lambda I - T)^{-1} | \lambda \in V - \{\infty\} \} \) is a bounded part of \( L(E, E) \). The set \( sp(T) = \hat{C} - r(T) \) is the \textit{spectrum} of \( T \). If \( sp(T) \not\subset \infty \) we say that \( T \) is \textit{regular}.

By an \textit{admissible set} we mean a directed (for \( \subset \)) set of closed parts of \( C \) whose union is \( C \), having a countable cofinal part and containing with \( A \subset C \) every closed part of \( A \). We denote below by \( C_0 \) and \( C_1 \) the admissible set of all compact parts of \( C \) and all closed parts of \( C \), respectively. Let \( C \) be an admissible set and \( T \) a closed linear mapping of \( D_T \subset E \) into \( E \). We say that \( T \) is a \textit{\( C \)-spectral operator} if there is a spectral family \( \mathcal{F} \) on \( C \) such that:

\begin{enumerate}
  \item[(D_1)] \( T \) commutes with \( \mathcal{F} \);
  \item[(D_II)] \( TU_{\mathcal{F}, f} \in L(E, E) \) for each \( f \in B^\infty(C) \) whose support is compact and belongs to \( C \);
  \item[(D_III)] \( sp(T_{\sigma}) \subset \sigma \) for every \( \sigma \in C \).
\end{enumerate}

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\( E \) barrelled means that every weakly bounded part of the dual space \( E' \) is equi-continuous; \( E \) quasi-complete means that every bounded closed part of \( E \) is complete. \( L(E, E) \) is the algebra of all linear continuous mappings of \( E \) into \( E \) endowed with the topology of uniform convergence on the bounded parts of \( E \).

\footnote{For a set \( A \subset C \) we denote by \( A^\sigma \) the closure of \( A \) in \( C \).}
For $\sigma \in \mathcal{S}(C)$ we denote by $T_\sigma$ the mapping $x \mapsto Tx$ of $D_T \cap E_\sigma$ into $E_\sigma$, where $E_\sigma = P_\mathcal{F}(\sigma)(E)$.

**Theorem 1.** Let $\mathcal{C}$ be an admissible set and $T$ a closed linear mapping of $D_T \subset E$ into $E$. Then there is at most one spectral family on $C$ satisfying $(D_1)$, $(D_{11})$ and $(D_{111})$.

For a $\mathcal{C}$-spectral operator $T$ we shall denote by $\mathcal{F}_T$ the unique spectral family on $C$ satisfying $(D_1)$, $(D_{11})$ and $(D_{111})$.

**Theorem 2.** Let $T$ be a $\mathcal{C}$-spectral operator. Then every $A \in L(E, E)$ commuting with $T$ commutes with $\mathcal{F}_T$.

Let now $\mathcal{C}$ be an admissible set of parts of $C$ and $\mathcal{F} = (m_{z, z'})_{z \in E, z' \in E'}$ a spectral family on $C$. Consider the following property concerning $\mathcal{F} : P\mathcal{C}$. Given $x \in E$, $x' \in E'$ there is $\sigma(x, x') \in \mathcal{C}$ such that the supports of the measures $m_{Q, z}$ are contained in $\sigma(x, x')$ for all $Q \in L(E, E)$ commuting with $\mathcal{F}$.

**Theorem 3.** Let $T$ be a $\mathcal{C}$-spectral operator and suppose that $\mathcal{F}_T$ has property $P\mathcal{C}$. Then $\text{sp}(T_\sigma) \subset \sigma^-$ for all $\sigma \in \mathcal{C}_1$.

**Theorem 4.** Let $T$ be a $\mathcal{C}$-spectral operator. Then $\mathcal{F}(\mathcal{F}_T)$ is compact.

2. We say that an operator $S \in L(E, E)$ is scalar if there is a spectral family $\mathcal{F} = (m_{z, z'})_{z \in E, z' \in E'}$ on $C$ of measures with compact support such that $\int \mathcal{C} dm_{z, z'} = \langle Sx, x' \rangle$ for all $x \in E, x' \in E'$; we write in this case $S = U_{\mathcal{F}_z}$. An operator $Q \in L(E, E)$ is quasi-nilpotent if $\lim_{n \to \infty} |\langle Q^n x, x' \rangle|^{1/n} = 0$ for all $x \in E, x' \in E'$.

**Theorem 5.** (5.1) Let $T \in L(E, E)$ be a $\mathcal{C}_0$-spectral operator and suppose that $\mathcal{F}_T$ has property $P\mathcal{C}_0$. Then $T = U_{\mathcal{F}_z} + Q$, where $Q$ is quasi-nilpotent, and $T$, $U_{\mathcal{F}_z}$, $Q$ commute. Further if $T = S + R$ where $S$ is scalar, $R$ quasi-nilpotent and where $T$, $S$, $R$ commute, then $S = U_{\mathcal{F}_z}$ and $R = Q$. (5.2) Let $\mathcal{F}$ be a spectral family on $C$ of measures with compact support and $Q$ a quasi-nilpotent operator commuting with $\mathcal{F}$. Then $T = U_{\mathcal{F}_z} + Q$ is a $\mathcal{C}_0$-spectral operator and $\mathcal{F} = \mathcal{F}_T$.

3. In what follows we denote by $\Phi$ an arbitrary directed family of closed barreled subspaces of $E$ having the properties: (i) the set $E_0 = \bigcup_{F \subset \Phi} F$ is dense in $E$; (ii) a linear mapping $T$ of $E_0$ into $E_0$ verifying the relations $T(F) \subset F$ for all $F \subset \Phi$ is continuous if $T(F)(T_F$ is the mapping $x \mapsto Tx$ of $F$ into $F$) is continuous for all $F \subset \Phi$; (iii) given

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*We denote by $S(\mathcal{F}_T)$ the closure in $C$ of the union of the supports of the measures belonging to $\mathcal{F}_T$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
x ∈ E and x′ ∈ E′ there is x₀ ∈ E₀ verifying the equations <Tx, x′> = <Tx₀, x′> for each T ∈ L(E, E) such that T(F) ⊂ F for all F ∈ Φ. Given Φ let LΦ(E, E) be the set of all T ∈ L(E, E) such that: (i) T(F) ⊂ F for all F ∈ Φ; (ii) T is regular for all F ∈ Φ and sp(T_F) ⊂ sp(T_{F′}) ⊂ sp(T) if F′, F″ ∈ Φ, F′ ⊂ F″. For T ∈ LΦ(E, E) we write A(T) = ∪_{F ∈ Φ} sp(T_F).

**Theorem 6.** If T ∈ LΦ(E, E) then sp(T) = A(T)^−.

**Theorem 7.** If T ∈ LΦ(E, E) then there exists a unique continuous representation J→j(T) of H(A(T)) into L(E, E) having the properties:

(7.1) J(T) = I;  (7.2) z(T) = T. Further j(T) ∈ LΦ(E, E) and sp(j(T)) = f(A(T))^− (f is an element in the equivalence class j).

Let T ∈ L(E, E) be a C₀-spectral operator. Suppose that σ₀ = (m_{z,x′}) z ∈ B, x′ ∈ B′ has property P₂(0) and let Φ = (E_φ)_φ ∈ Φ). Then Φ has the properties (i), (ii), (iii) and T ∈ LΦ(E, E). Moreover:

**Theorem 8.** The operator j(T) is C₁-spectral for each j ∈ H(A(T)) and

\[ (1) \quad \langle f(T)x, x′ \rangle = \sum_{j=0}^{∞} \frac{1}{j!} \int_{C} f^{(j)} dm_{x,x′}, \quad \text{for } x ∈ E, x′ ∈ E′, \]

where Q is the quasi-nilpotent part of T. The series (1) converges absolutely and uniformly for given x ∈ E and x′ ∈ A (A is an arbitrary equicontinuous part of E′).

4. Let C be an admissible set, (σ(n)) an increasing sequence of compact parts belonging to C whose union is C, T : D_T → E a C₁-spectral operator, σ₀ = (m_{z,x′}) z ∈ B, x′ ∈ B′, and E₀ = ∪E_σ(n). Let T₀ be the restriction of T to E₀ ⊂ D₀ and σ₀ = (m_{z,x′}) z ∈ E₀, x′ ∈ E₀. Here E₀ is endowed with the topology, inductive limit of the topologies of the subspaces E_σ(n) of E, and, for x ∈ E_σ(n) ⊂ E₀ and x′ ∈ E′₀, m_{z,x′} = m_{z,x′}, if y′ ∈ E′ is such that x₀_y = y₀y′. If y′ ∈ E′ has property P₂(0) then j(T) has property P₂(0). Further A ∈ L(E, E) commutes with T if and only if A(E₀) ⊂ E₀ and A_{B₀} commutes with T₀. Also T is "scalar" if and only if T₀ is scalar; if T is "scalar" and f is such that φ_{E}(n)f ∈ B_{n}(C) for all n then f(T)=f(T₀).

**Theorem 9.** (9.1) T₀ is a C₁-spectral operator, σ₀ = σ₀ and σ₀ has property P₂(0) (Σ is the smallest admissible set containing (σ(n))).

(9.2) T is the closure of T₀. (9.3) sp(T₀) = S(σ₀)^−.

Further A ∈ L(E, E) commutes with T if and only if A(E₀) ⊂ E₀ and A_{B₀} commutes with T₀. Also T is "scalar" if and only if T₀ is scalar; if T is "scalar" and f is such that φ_{E}(n)f ∈ B_{n}(C) for all n then f(T)=f(T₀).

4 For the definition of H(A), A ⊂ C (endowed with the "van Hove topology"), see for instance [5] (where A is supposed compact) and [4, pp. 255-256].
5. The subject matter of this note has been suggested by [2] and by [1; 3]. The Theorems 1, 2, 5 and 8 are essentially generalizations of the corresponding results in [2; 3]. The results of paragraph 4 show some of the relations between the unbounded spectral operators defined in [1] and the (everywhere defined continuous) spectral operators defined above. The definition of the spectrum and of the quasi-nilpotent operator were suggested by definitions given in [5; 6], respectively.

REFERENCES


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