FIXED POINTS OF MULTIPLE-VALUED TRANSFORMATIONS

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A *multiple-valued transformation* T from a space X to a space Y is a function assigning to each point x of X a nonempty closed subset $T(x)$ of Y. The *graph* of T comprises those points (x, y) in the topological product XY for which y belongs to $T(x)$. All spaces to be considered shall be compact metric and all transformations T shall be upper semi-continuous, meaning that their graphs are closed, hence compact, subsets of XY.

When the domain X and the range Y of T coincide, a *fixed point* of T is defined to be a point x which belongs to its image set $T(x)$. The fixed points correspond to those points in the product XX which belong to the intersection of the graph of T with the diagonal of XX. In the special case that X is an orientable n-manifold the diagonal carries an n-cycle D. If, in addition, each neighborhood of the graph contains a representative cycle of an n-cycle homology class Γ whose intersection number with D is not zero, then the diagonal must meet the graph of T, so that under the assumptions made at least one fixed point must exist. To each n-cycle class Γ having representative cycles in each neighborhood of the graph corresponds an endomorphism $T_{*\Gamma}$ of the homology group $H(X)$ of X, determined as follows: starting with any p-cycle γ, form in the product XX the upright cylinder $\gamma \times X$, intersect the cylinder with Γ and project the intersection laterally into X to obtain finally $T_*(\gamma)$. If the homology of X uses a field as coefficient group, then the endomorphisms $T_{*\Gamma}$, constitute a vector space $*(T)$. For a single-valued transformation τ, the space $*(\tau)$ comprises scalar multiples of the conventional endomorphism τ_* induced by τ. The definition of $*(T)$, here described for manifolds only, has been extended to an arbitrary A.N.R., using singular homology, by Lefschetz [8] and to an arbitrary compact metric space, using Čech homology, by O'Neill [11].

The Lefschetz fixed point theorem, extended to multiple-valued transformations, assumes the following form: Let X be an A.N.R. and let T be an upper semi-continuous multiple-valued transformation of X into itself. Then either T has a fixed point or else the equa-

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tion $\sum(-1)^p$ trace $T_{*\Gamma_p}=0$ holds for each $T_{*\Gamma}$ in $*(T)$. $T_{*\Gamma_p}$ denotes the endomorphism which $T_{*\Gamma}$ induces among the p-cycles of $H(X)$. It is obvious that before any application of the Lefschetz theorem can be made the space $*(T)$, or at least an appropriate element of it, must be known. Eilenberg and Montgomery [**1**] showed that if the image sets $T(x)$ are acyclic (using Vietoris homology) and X is an acyclic A.N.R. then $*(T)$ contains an endomorphism which maps the 0-cycles nontrivially, so that the Lefschetz theorem can be applied to show that a fixed point must exist. Their result extended earlier results of Kakutani [**7**], who assumed X to be a ball and $T(x)$ convex and of Wallace [**16**], who assumed X to be a tree and $T(x)$ connected. All the above theorems assume in effect that $T(x)$ is a swollen point so that T can be thought of as a "sloppy" single-valued transformation. Other authors have demonstrated fixed point theorems for multiple-valued transformations under different continuity assumptions on T (see [**4**; **5**; **10**; **11**; **12**; **13** and **17**]).

The theorem of Kakutani mentioned above is motivated by its application to game theory. In what follows two other applications of multiple-valued function fixed point theorems are presented. The second application uses the Lefschetz theorem in a situation where the image sets are not acyclic.

The first application is to the problem of finding coincidences of two mappings f, g proceeding in the same direction from one space to another, i.e., points where $f(x)=g(x)$. To elucidate the method it shall be applied to the following simple example, typical of a wide class. Let h be a simplicial mapping of an n-sphere S^n into the real line. It was shown by Johnson [**6**], using a method due to Sorgenfrey [**15**], and by Sonneborn [**14**] that it is possible to find a pair of antipodal points of S^n belonging to the same component of a level line of h.[1] Expressed intuitively, the result says that there are two antipodal points on the moon such that one can walk from one to the other, always staying at the same altitude. The proof outlined below derives this result from the theorem of Wallace cited above.

Let K^1 be the identification space obtained from S^n whose points are the components of the level lines of h and let f denote the identification map, $f: S^n \to K^1$. K^1 is a tree. If, further, α denotes the antipodal mapping of S^n onto itself, then a solution of the coincidence

[1] Actually, Johnson and Sonneborn assumed only continuity of the real-valued function on the sphere. In this case the identification space is a dendrite and one can attempt to carry the argument through in the same way, but the various extensions of Wallace's theorem to a dendrite require continuity assumptions stronger than those available here. The continuous case can, however, be derived from the simplicial one by an approximation argument.

equation $f(x)=f\alpha(x)$ gives the desired antipodal pair. The inverse of f is an upper semi-continuous multiple-valued transformation from K^1 to S^n, mapping each point k onto a connected set $f^{-1}(k)$; likewise the composite transformation $f\alpha f^{-1}$ maps each point of the tree K^1 into a connected subset of K^1, so that by the theorem of Wallace, there must exist a fixed point $k_0 \in f\alpha f^{-1}k_0$; any point x in $f^{-1}k_0$ satisfies the required equation $f(x)=f\alpha(x)$.

Analysis of the above proof reveals that considerable weakening of the hypotheses is possible. S^n need not be a sphere: the identification space K^1 will be a tree if we assume only that S^n is a connected polyhedron P whose first Betti number vanishes. The combination $f\alpha$ could be any continuous function g from P to K^1, in particular, therefore, α need not be antipodal. The existence of a solution to the coincidence equation $f(x)=g(x)$ could not have been shown by tools which depend only on the homotopy classes of f, g: $P\to K^1$, because f and g can be deformed into unequal constant mappings (unless K^1 is a single point). The usual theory of fixed points, using only methods, such as the induced homology homomorphisms, which depend only on homotopy class, extends successfully to coincidences only when the image space is a manifold [**3**] or when f and g proceed in opposite directions between the two spaces [**8**; **9**]. The application of the multiple-valued transformation concept to coincidence theory assumes the following general form: of the two mappings f, g from X to Y, suppose that f is onto and let $f^{-1}_{*\Gamma}$ belong to the space $*(f^{-1})$ of endomorphisms induced by the multiple-valued transformation f^{-1}. If $\sum(-1)^p$ trace $g_{*p}f^{-1}_{*\Gamma p}$ is not zero, then f and g have a coincidence. The assumptions made on f generalize those properties of the identity essential to the success of the theory of fixed points of single-valued mappings.

The second application shows how the theory of fixed points of multiple-valued transformations can be used to prove the existence of periodic solutions of ordinary differential equations. As before, the discussion is based on a simple but typical example. Let the solid torus T be the product $E^n\times S^1$ of an n-ball and a circle and let T bear a nowhere vanishing differentiable vector field which points into T on its boundary, so that the field's positive semi-orbits lie in T. Under what conditions on the field can the existence of a closed trajectory be established? The classical approach to this question, dating from Poincaré, is that of the surface of section. A surface of section S is a hypersurface whose tangent plane at each point never contains the vector of the field. If, in addition, for each point P of S the positive semi-orbit starting out from P returns to S and the assignment to P of the first such return defines a continuous mapping of S into itself, then a fixed point of this mapping corresponds to a closed trajectory of the field. Unfortunately, even if a surface of section can be found, it may happen that the positive semi-orbits from some points of S never return to S, so that the desired mapping of S into itself can not be constructed. For $T=E^3\times S^1$ the author has constructed an example [2] where a surface of section exists but the field has no closed trajectories.

The above considerations suggest that the crucial assumption of the surface of section method is that the semi-orbits continue to move about T. In order to express this assumption exactly some notation is required. Let $\tilde{T}=E^n\times R$ denote the universal covering of T and let the real number θ be a coordinate for R, so that reduction mod 2π of θ projects $\tilde{T}$ onto T. The vector field on T induces a vector field on $\tilde{T}$ whose orbits lie over the orbits in T. For each point P of T and each time t, we may define $\int_{C(t,p)}d\theta$ to be the integral of the 1-form $d\theta$ over the semi-orbit $C(t, p)$ proceeding from P to the point arrived at after time t (regarding the field as a velocity field). $\int_{C(t,p)}d\theta$ could also be regarded as the difference between the final and initial values of θ for any trajectory in $\tilde{T}$ covering $C(t, p)$. We may now state the following theorem, in which no assumption whatever is made regarding the existence of a surface of section.

THEOREM. *Let the solid torus T bear a vector field as above and suppose that $\int_{C(t,p)}d\theta\to+\infty$ uniformly in P as $t\to\infty$. Then there exists a closed trajectory of the field.*

The proof outlined below shows that by using the multiple-valued transformation concept, the cross-section $\theta=0$ can be made to play the rôle of the surface of section. A multiple-valued transformation ρ from the cross-section ball $\theta=0$ in $\tilde{T}$ to the ball $\theta=2\pi$ can be constructed by assigning to each point P of $\theta=0$ the intersection of $\theta=2\pi$ with the positive semi-orbit in $\tilde{T}$ which starts at P. The ball $\theta=0$ is eventually carried past $\theta=2\pi$ by the field; for the sake of exposition, we assume that $\rho(P)$ is a finite set of points.[2] Each point of $\rho(P)$ may be assigned an index $+1$, -1 or 0 according as the trajectory crosses the disk $\theta=2\pi$ from the side $\theta<2\pi$ to $\theta>2\pi$, from $\theta>2\pi$ to $\theta<2\pi$ or glances off the disk $\theta=2\pi$, remaining on one side. To each

[2] The assumptions made on ρ can be justified by arguing as follows. The assignment to each point of the ball $\theta=0$ of the semi-orbit starting from P and continuing until a time when all the orbits have gone past $\theta=2\pi$ is a multiple-valued function from $\theta=0$ to $\tilde{T}$. The graph of this function is an $(n+1)$-chain in $E^n\times\tilde{T}$. If this $(n+1)$-chain is replaced by a simplicial approximation in general position with respect to the $2n$-chain defined by $\theta=2\pi$, the intersection of these two chains is a relative n-cycle Γ lying in a neighborhood of the graph of ρ which defines ρ_* as described previously.

point P of $\theta=0$, regarded as an elementary 0-cycle, is assigned the 0-cycle $\rho_*(P)$ in the disk $\theta=2\pi$ obtained by indexing the image points as above; the sum of the indexes is always $+1$, so that the image cycle does not represent a trivial homology class. The ρ_* images of higher dimensional chains may be obtained in a similar way. If the disk $\theta=0$ is now identified with the disk $\theta=2\pi$ by equating points equivalent in T, ρ defines an upper semi-continuous multiple-valued transformation of a disk into itself and the element ρ_* of $*(\rho)$ described above gives the value $+1$ to the Lefschetz formula, so that ρ must have a fixed point; this fixed point defines a closed trajectory of the field in T, as required.

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