SIMULTANEOUS RATIONAL APPROXIMATIONS TO ALGEBRAIC NUMBERS

BY L. G. PECK

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Let $K$ be an algebraic number field of degree $n+1$ over the rationals. The conjugates $K^{(0)}, K^{(1)}, \ldots, K^{(n)}$ are arranged so that $K^{(0)}, K^{(1)}, \ldots, K^{(r)}$ are real and

$$K^{(r+k)} = \overline{K^{(r+k)}}, \quad (k = 1, 2, \ldots, s).$$

Here $r+2s=n$. It will be assumed throughout that $r \geq 0$, so that $K^{(0)}$ is real. Numbers in $K$ are denoted by Greek letters, superscripts being used for the corresponding conjugates. We shall frequently omit the superscript $(0)$; this identification of $K$ with $K^{(0)}$ will cause no confusion. Trace and norm of elements of $K$ are denoted by $S$ and $N$, respectively.

Let $\beta_0, \ldots, \beta_n$ be elements of $K$ which are linearly independent over the rationals. It is well known that infinitely many sets of rational integers $(q_0, q_1, \ldots, q_n)$ can be found satisfying,

1. $q_0 > 0$, $\text{g.c.d.}(q_0, q_1, \ldots, q_n) = 1,$

and (omitting the superscript $(0)$)

2. $$\frac{\beta_j - q_j}{\beta_0 - q_0} < C q_0^{-1/n}, \quad (j = 1, \ldots, n),$$

with the constant $C=1$. It will be shown here how to determine all solutions of (1), (2). From this will be deduced not only the known fact that if $C$ is too small (2) has no solutions, but also the hitherto unknown result that the sharper inequalities

3. $$\begin{align*}
|q_0 \beta_j - q_j \beta_0| &< C q_0^{-1/n} (\log q_0)^{-1/(n-1)}, \\
|q_0 \beta_n - q_n \beta_0| &< C q_0^{-1/n},
\end{align*}$$

$(j = 1, \ldots, n - 1)$, have infinitely many solutions.

This result sharpens some of the conclusions of Cassels and Swinnerton-Dyer (I), but does not furnish any further evidence for or against the conjecture of Littlewood which is considered in their paper.

A number of interesting problems can be raised in connection with (3). In one direction it can be asked whether $n-1$ of the inequalities (2) can be improved with factors which are not all the same; e.g.,
one might conjecture that we can find infinitely many solutions of
the inequalities

\[ |q_0 \beta_j - q_j \beta_0| < C q_0^{-1/n} f_j(q_0), \]

\[ |q_0 \beta_n - q_n \beta_0| < C q_0^{-1/n}, \]

with \( f_1(q_0) \cdots f_{n-1}(q_0) = \log q_0 \) and \( f_j(q_0) \geq 1 \) \( (j = 1, \ldots, n-1) \).

A much more difficult set of problems is in the direction of the
Thue-Siegel-Roth theorem, in which one tries to specify the functions
\( f_j \) in such a way that the corresponding inequalities have at most a
finite number of solutions. In view of Roth’s theorem one might con­
jecture that \( f_j = q_0^\epsilon \) would have the indicated effect, but this is by
no means obvious.

The numbers \( \beta_0, \ldots, \beta_n \) form the basis of a module \( M \). Denote
by \( R \) the set of all integers \( p \) in \( K \) such that \( p/3 \) is in \( M \) whenever
\( 0 \) is in \( M \). Clearly \( R \) is a ring. By the Dirichlet theory of units, we may
find a basis \( \epsilon_1, \ldots, \epsilon_r+1 \) of the units in \( R \). Since the only roots of
unity in \( K \) are \( \pm 1 \) (because \( K(0) \) is real) every unit \( \epsilon \) in \( R \) is uniquely
expressible in the form

\[ \epsilon = \pm \epsilon_1 \epsilon_2 \cdots \epsilon_{r+1} \cdot \]

Let \( C_1 = \max_{i, j=1, \ldots, r+s} |\log |\epsilon_j^{(i)}|| \). Then, for any real number \( T \)
we can find integers \( g_1, \ldots, g_{r+s} \) such that

\[ -n^{-1} T - \frac{1}{2} C_1 \leq \sum_{j=1}^{r+s} g_j |\log \epsilon_j^{(i)}| < -n^{-1} T + \frac{1}{2} C_1, \]

\[ (i = 1, \ldots, r+s), \]

and, since \( N(\epsilon_j) = 1 \),

\[ T - \frac{n}{2} C_1 < \sum_{j=1}^{r+s} g_j |\log \epsilon_j| \leq T + \frac{n}{2} C_1. \]

Using (4) with the sign chosen so that \( \epsilon > 0 \), we obtain

\[ |\epsilon^{(i)}| < C_2 n^{1/n} \]

with a constant \( C_2 = e^{C_1} \) which depends only on the ring \( R \). A unit
\( \epsilon > 1 \) which satisfies (5) will be called dominant. We have proved that
for every real \( T > 1 \) there is a dominant unit \( \epsilon \) satisfying \( T \leq |\epsilon| < C_2 T \).

The elements \( \delta \) of \( K \), such that \( S(\delta \beta) = a \) rational integer for every
\( \beta \) in \( M \), form another module \( D \). A basis \( \delta_0, \delta_1, \ldots, \delta_n \) of \( D \) is obtained
by solving the equations

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(6) \[ S(\beta, \delta_j) = \begin{cases} 1, & (i = j) \\ 0, & (i \neq j) \end{cases} \] (i, j = 0, \ldots, n).

Because of the discreteness of \( D \), there is, among the nonzero elements of \( D \), one whose norm has minimal absolute value; this minimal norm will be denoted by \( v \). Note also that if \( \rho \) is in \( R \) and \( \delta \) in \( D \) then \( \rho \delta \) is in \( D \).

Choose \( \delta = a_0 \delta_0 + a_1 \delta_1 + \cdots + a_n \delta_n \) in \( D \) so that \( \delta \beta_0 > 0 \) and g.c.d.\( (a_0, a_1, \ldots, a_n) = 1 \). If \( \epsilon \) is a unit in \( R \) and \( \epsilon \delta = q_0 \delta_0 + q_1 \delta_1 + \cdots + q_n \delta_n \) we must have g.c.d.\( (q_0, \ldots, q_n) = 1 \). If g.c.d.\( (q_0, \ldots, q_n) = q_0 \), it is clear that \( q^{-1} \epsilon \delta \) is in \( D \), whence \( \epsilon^{-1} q^{-1} \epsilon \delta = q^{-1} \delta \) is in \( D \) and \( q \) divides g.c.d.\( (a_0, \ldots, a_n) = 1 \).

As defined above, we have

(7) \[ q_k = S(\epsilon \delta \beta_k), \quad (k = 0, \ldots, n). \]

Thus, if we assume that \( \epsilon \) is dominant, we have

\[
| q_k \beta_0 - q_0 \beta_k | = \left| \sum_{j=1}^{n} (\beta_k^{(j)} \beta_0 - \beta_k \beta_0^{(j)}) \delta^{(j)} \right| 
< C_3 \epsilon^{-1/n}, \quad (k = 1, \ldots, n),
\]

while

\[
| q_0 - \epsilon \delta \beta_0 | = \left| \sum_{j=1}^{n} \epsilon^{(j)} \delta^{(j)} \beta_0^{(j)} \right| < C_4 \epsilon^{-1/n}.
\]

The last two inequalities imply (2). The constants \( C_3, C_4, C \) depend on \( a_0, \ldots, a_n, \) but we may remove this dependence if the choice of \( \delta \) is made from a fixed bounded region.

Suppose conversely that (1) and (2) hold (with some \( C > 0 \)). Define \( \xi = q_0 \delta_0 + \cdots + q_n \delta_n \), so that \( \xi \) is in \( D \). We have

\[
\xi^{(i)} = \frac{1}{\beta_0} \sum_{j=1}^{n} (q_j \beta_0 - q_0 \beta_j) \delta_j^{(i)} + \frac{q_0}{\beta_0} \sum_{j=0}^{n} \beta_j \delta_j^{(i)}, \quad (i = 0, \ldots, n).
\]

It follows easily from (6) that the last sum is 1 or 0 according as \( i = 0 \) or \( i \neq 0 \). Thus

\[
\left| \xi - \frac{q_0}{\beta_0} \right| = C q_0^{-1/n} \sum_{j=1}^{n} | \delta_j |,
\]

while

\[
| \xi^{(i)} | < C q_0^{-1/n} \sum_{j=1}^{n} | \delta_j^{(i)} |, \quad (i = 1, \ldots, n).
\]
Choose a dominant unit \( \epsilon \) such that \( \left| q_0/\beta_0 \right| \leq \epsilon < C_2 \left| q_0/\beta_0 \right| \) and set \( \delta = \pm \epsilon^{-1/2} \) with the sign chosen so that \( \delta > 0 \). Then
\[
0 < \delta < 1 + C \left| \beta_0 \right|^{-1} q_0^{-1/2} \sum_{j=1}^{n} |\delta_j|,
\]
while
\[
0 < |\delta^{(i)}| < C \left| \epsilon^{(i)} \right|^{-1} q_0^{-1/2} \sum_{j=1}^{n} |\delta_j^{(i)}|.
\]
Thus
\[
0 < |N(\delta)| < C^n \epsilon q_0^{-1} \prod_{i=1}^{n} \sum_{j=1}^{n} |\delta_j^{(i)}| (1 + O(q_0^{-1/2}))
\]
\[
< (C C_2)^n C_4,
\]
where \( C_4 \) depends only on \( \beta_0, \ldots, \beta_n \). It follows that \( \delta \) is an element of \( D \) which lies in a bounded region (which will be vacuous if \( C \leq v^{1/2}/C_4 C_4^{1/2} \)) and that the \( q_k \) are given by (7).

The proof of (3) is based on a special choice of \( \delta \) in (7) together with a sharper form of (5) for a certain infinite set of dominant units.

To obtain the latter, let
\[
\epsilon_k^{(j)} = \begin{cases} \epsilon_k^{-1/2} \epsilon^{(j)} e^{j \phi_{jk}} e^{j \psi_{jk}}, & (j = 1, \ldots, r), \\
\epsilon_k^{-1/2} \epsilon^{(j)} e^{j \phi_{jk} + \epsilon \psi_{jk}}, & (j = r + 1, \ldots, r + s),
\end{cases}
\]
where \( \phi_{jk} \) and \( \psi_{jk} \) are real and \( \epsilon^{jb} = \pm 1 \).

If the dominant unit \( \epsilon \) is given by (4) we have
\[
\left| \sum_{k=1}^{r+s} \epsilon_k g_k \right| = |\log | \epsilon^{1/2} \epsilon^{(j)} | < C_1, \quad (j = 1, \ldots, r + s).
\]

Also, we can find rational integers \( h_j \) such that
\[
(2\pi)^{-1} \arg \epsilon^{(j)} = \sum_{k=r+1}^{r+s} \psi_{jk} g_k + h_j \leq 1/2.
\]

Now there are at least \( M+1 \) distinct dominant units \( \epsilon \) in the interval \( 1 \leq \epsilon < e^{(M+1)} C_1 \). By the well known schubfachprinzip of Dirichlet we may therefore find two—call them \( \eta \) and \( \theta \)—such that \( 1 \leq \theta < \eta < e^{(M+1)} C_1 \),

\[
\left| \log \left| \eta^{1/2} \eta^{(j)} \right| - \log \left| \theta^{1/2} \theta^{(j)} \right| \right| < 2C_1/M^{(n-1)}, \quad (j = 2, \ldots, r + s)
\]
and
\[
\pi^{-1} |\arg \eta^{(j)} - \arg \theta^{(j)}| \leq M^{-1/(n-1)}, \quad (j = r + 1, \ldots, r + s).
\]
Thus the unit $e = \eta / \theta$ satisfies $1 < \epsilon < C_2^n T$ (where $T = e^{\mu n C_1}$) and
\[
| \log | \epsilon^{1/n} e^{(j)} | | < 2(C_1^n n/\log T)^{1/(n-1)}, \quad (j = 2, \ldots, r + s),
\]
\[
| \arg e^{(j)} | \leq 2\pi (C_1^n n/\log T)^{1/(n-1)}, \quad (j = r + 1, \ldots, r + s).
\]
Moreover, since
\[
\sum_{j=1}^{r} \log | \epsilon^{1/n} e^{(j)} | + 2 \sum_{j=r+1}^{r+s} \log | \epsilon^{1/n} e^{(j)} | = 0,
\]
we have also
\[
| \log | \epsilon^{1/n} e^{(1)} | | < 2(n - 1)(C_1^n n/\log T)^{1/(n-1)}.
\]
It follows that
\[
e^{(j)} = | \epsilon^{(j)} | \exp(i \arg e^{(j)}) = \epsilon^{-1/n}(1 + O(\log T)^{-1/(n-1)}),
\]
\[(j = 1, \ldots, r + s),
\]
which is the required refinement of (5).

If we choose $\delta = \delta_n$ in (7) and make use of (6) and (9) we can improve (8) as follows:
\[
| q_k^* \beta_0 - q_k^* \beta_k | = \epsilon^{-1/n} \left( \left| \sum_{j=1}^{n} (\beta_k^{(j)} \beta_0 - \beta_k \beta_0^{(j)}) \delta_n^{(j)} \right| + O(\log T)^{-1/(n-1)} \right)
\]
\[
= \begin{cases} 
O(\epsilon^{-1/n}(\log T)^{-1/(n-1)}) & (k = 1, \ldots, n - 1) \\
O(\epsilon^{-1/n}) & (k = n).
\end{cases}
\]
This, together with $1 < \epsilon < C_2^n T$, implies (3).

**Reference**


**Boston, Massachusetts**