INTEGRATION WITH RESPECT TO OPERATOR-
VALUED FUNCTIONS

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1. Introduction. Let $J$ be a compact subinterval of the real line. N. Wiener [7] has introduced the Banach algebra $W_p(J)$ of all complex-valued functions $f$ such that $V_p(f) \neq \infty$, where

$$V_p(f) = \sup \left( \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p},$$

the supremum being taken over all finite partitions of $J$ (see §7). We shall construct a family of continuous homomorphisms of the Banach algebra $W_p(J)$; this connects with the theory of multipliers of Fourier series (see §4). Our basic problem is to integrate (in the uniform operator-topology) with respect to functions that are not of bounded variation.

Given a fixed measurespace $(a, \mathcal{A}, \mu)$, let $\mathcal{E}_r$ denote the Banach algebra of all continuous endomorphisms of $L_r(a, \mathcal{A}, \mu)$; the relation $1 < r < \infty$ is implied throughout. Let $E_r$ be a function on $J$ which assumes its values in $\mathcal{E}_r$, and let $f$ belong to the class $D(J)$ of all simply-discontinuous, complex-valued functions. The following expression

$$\int f(\lambda) \cdot dE_r(\lambda)$$

will denote what T. H. Hildebrandt [1, p. 273] calls the “modified Stieltjes integral”; it is the limit of a certain net of Stieltjes sums (this net is directed as in the Pollard-Moore integral [1, p. 269]). The word “limit” here implies convergence in the norm-topology of $\mathcal{E}_r$. It is not hard to show that the integral (1) converges when $E_r$ is of bounded variation; this situation is most familiar in the case $r = 2$, when $E_r$ is a resolution of the identity in the Hilbert space $L_2(a, \mathcal{A}, \mu)$. Henceforth, we will allow the possibility that $E_r$ not be of bounded variation (this possibility becomes a fact in Theorem D below).

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2 That is, having on $J$ at most discontinuities of the first kind.

3 In the sense of Hille-Phillips [2, p. 59]. Bounded variation is a less restrictive condition than the bounded semi-variation hypothesis required in certain integration theories (e.g., Bartle’s article in the Studia Math. vol. 15 (1956) pp. 337–352).
2. **Motivation.** Suppose that $r \neq 2$. Some integrators $E_r$ have the following property: there exists no spectral measure $M$ such that

$$\int \lambda \cdot M(d\lambda) = (E_r) \int \lambda \cdot dE_r(\lambda),$$

although the integral on the right-hand side converges.

3. **An operational calculus.** Let $L^0(a, \alpha, \mu)$ be the class of all simple functions. If $T \subset \mathbb{C}_2$ we write

$$(2) \quad |T|_r = \sup \{\|Tx\|_r : x \in L^0(a, \alpha, \mu) \text{ and } |x|_r \leq 1\};$$

it is clear that the eventuality $|T|_r \neq \infty$ implies the existence of the continuous extension (denoted $T_r$) of $T$ from $L^0(a, \alpha, \mu)$ to $L_r(a, \alpha, \mu)$.

Suppose that $E$ is a resolution of the identity in $L_2(a, \alpha, \mu)$ such that

$$\infty \neq \sup_{\lambda \in \Lambda} |E(\lambda)|_s \quad \text{whenever} \ 1 < s < \infty.$$  

If $\lambda \in J$, then $E(\lambda) \subset \mathbb{C}_2$ and $|E(\lambda)|_r \neq \infty$, whence $E(\lambda)_r \subset \mathbb{C}_r$ (here $E(\lambda)_r$ again denotes the extension of $E(\lambda)$ from $L^0(a, \alpha, \mu)$ to $L_r(a, \alpha, \mu)$). Accordingly, we may define on $J$ a function $E_r$ by means of the relation $E_r(\lambda) = E(\lambda)_r$. Finally, let $I(\phi)$ denote the open interval with endpoints $2\phi/(\phi \pm 1)$. Under these circumstances, it can be proved that:

- if $1 \leq \phi < \infty$ and $r \in I(\phi)$, then each integral of the family

$$\left\{(E_r) \int f(\lambda) \cdot dE_r(\lambda) : f \in W_\phi(J)\right\}$$

converges in the norm-topology of $\mathbb{C}_r$.

It is notable that, if $\infty > \phi > q > 1$, then

$$\{2\} \subset I(\phi) \subset I(q) \subset I(1) = \{\lambda : 1 < \lambda < \infty\},$$

$$D(J) \supset W_\phi(J) \supset W_q(J) \supset W_1(J) = \{\text{bounded variation}\}.$$  

In other words: as the range of $r$ expands from the Hilbert-space case $\{2\}$ to comprise the whole interval $(1, \infty)$, then $W_\phi(J)$ contracts into the class $W_1(J)$ consisting of all functions of bounded variation.

There exists a well-known bijection $\{T \rightarrow E^T\}$ of the class $\mathcal{E}$ (of all self-adjoint members of $\mathbb{C}_2$) into the class of all resolutions of the identity [6, p. 174 and p. 176]. Suppose $T \subset \mathcal{E}$, and let $J$ be an interval that contains the spectrum of $T$. It will be convenient to write

$$f(T_r) = (E_r) \int f(\lambda) \cdot dE^T_r(\lambda).$$
An application of the Spectral Theorem shows easily that:

\[ \text{if } f(\lambda) = \sum \alpha_n \cdot \lambda^n \text{ is a polynomial, then } f(T_r) = \sum \alpha_n \cdot (T_r)^n. \]

**Theorem A.** Suppose that \( T \in \mathfrak{T} \) and let condition (v) be satisfied when \( E \) is replaced by \( E^r \). If \( 1 \leq p < \infty \) and \( r \in I(\phi) \), then the mapping \( \{ f \rightarrow f(T_r) \} \) is a continuous homomorphism of the Banach algebra \( W_p(J) \) into \( \mathfrak{E}_r \).

4. **Two applications to the theory of multiplier transformations.**

Consider a complete orthonormal system \( \{ \Phi_n : n \in \mathfrak{A} \} \); accordingly, \( \mathfrak{A} \) is denumerable. In this paragraph, \( \mathfrak{A} \) consists of all subsets of \( \mathfrak{A} \), while \( \mu \) is taken to be counting-measure; thus \( L_r(\mathfrak{A}, \mathfrak{A}, \mu) \) becomes the sequence space usually denoted \( l_r \), and \( L^0(\mathfrak{A}, \mathfrak{A}, \mu) \) is now the class \( l^0 \) of all sequences that vanish off finite subsets of \( \mathfrak{A} \). If \( x \in l^0 \), then \( f_x(x) \) will denote the sequence of Fourier coefficients of the function \( f \mapsto x^{- \infty} \), where

\[ (f \cdot x^{- \infty})(\lambda) = f(\lambda) \cdot \sum_{n \in \mathfrak{A}} x_n \cdot \Phi_n(\lambda) \quad (\lambda \in J). \]

Let \( f_x \) be the mapping \( \{ x \mapsto f_x(x) \} \) defined on \( l^0 \). Hirschman [3] calls \( f_x \) a “multiplier transformation.” An important problem in the theory of multiplier transformations is to find conditions on \( f \) which will insure that \( |f_x | \neq \infty \).

**First application.** Let \( \{ \Phi_n : n \in \mathfrak{A} \} \) be the system of normalized Legendre polynomials on \( J = [-1, 1] \), and denote by \( T \) the member \( \Delta \) of \( \mathfrak{T} \) that is defined in \[4, (2)\]. The article \[4\] shows that condition (v) is satisfied when \( E \) is replaced by \( E^r \). Suppose \( 1 \leq p < \infty \) throughout. Our theory shows that

\( i \) if \( r \in I(\phi) \) and \( f \in W_p(J) \), then \( |f_x | \neq \infty \);

consequently, \( f_x \) has a continuous extension \( f_x \), from \( l^0 \) to \( l_r \). In fact, we can prove

**Theorem B.** If \( r \in I(\phi) \) and \( f \in W_p(J) \), then \( f(T_r) \in \mathfrak{E}_r \).

**Second application.** From now on, \( \{ \Phi_n : n \in \mathfrak{A} \} \) will be the trigonometric system; \( \mathfrak{A} \) is now the integer group and \( J = [0, 1] \). Originally proved by Stečkin in the case \( p = 1 \), property (i) was discovered by Hirschman [3]; his proof is based on Stečkin's result. Theorem B is also valid in the present context, the operator \( T \) being now the Hilbert transformation defined for all \( x \) in \( l_2 \) by the relation

\[ (Tx)_n = \sum_{k \in \mathfrak{A}} x_k \cdot \frac{i}{2\pi(n-k)} \quad (k \neq n) \]
for each $n$ in $a$. On the strength of Theorem A, we can prove Theorem B directly from the following well-known property: $|T|_s \neq \infty$ whenever $1 < s < \infty$.

**Theorem D.** Let $T_r$ be the unitary shift operator defined (for all $x$ in $l_r$) by the relation $T_r x = \{n \mapsto x_{n+1}\}$; there exists a function $E_r$ on $J$ to $\mathbb{C}$, such that

$$T_r = (\mathbb{C}_r) \int e^{-2\pi i \lambda \cdot} \cdot dE_r(\lambda),$$

although $E_r$ is not of bounded variation.$^3$

5. Hölder-type inequalities and the variation-norm. We now return to the general setting of §3; once again, $(a, \alpha, \mu)$ is an arbitrary measure space, and the integrator $E$ is a resolution of the identity satisfying (v). If $x, y \in L^0(a, \alpha, \mu)$, then the relation

$$E_{x,y}(\lambda) = \int_a^\lambda y \cdot E(\lambda) x \cdot d\mu$$

defines a complex-valued function $E_{x,y}$. The variation-norm is defined as follows:

$$V_q(E) = \sup \{ V_q(E_{x,y}) : x \in U_r \text{ and } y \in U_{r'} \},$$

where $U_r = \{ z \in L^0(a, \alpha, \mu) : ||z||_r \leq 1 \}$ and $r' = r/(r-1)$. When $f \in D(J)$ and $r = 2$ it is easy to verify the familiar inequality

$$(ii) \quad \left| (\mathbb{C}_r) \int f(\lambda) \cdot dE_r(\lambda) \right|_r \leq V_1(E)_{r'} ||f||_\infty,$$

where $||f||_\infty = \sup \{ ||f(\lambda)|| : \lambda \in J \}$.

Suppose $1 < p < \infty$ and $r \in I(p)$. Our approach involves establishing the existence of a number $q > 1$ such that $q^{-1} + p^{-1} > 1$ and

$$(ii^*) \quad \left| (\mathbb{C}_r) \int f(\lambda) \cdot dE_r(\lambda) \right| \leq c_{r,p} \cdot V_q(E)_{r'} (||f||_\infty + V_p(f)) < \infty$$

(where $c_{r,p}$ is independent of $f$ and $E$), for each $f$ in $W_p(J)$. This is closely related to a theorem of Love and L. C. Young [5]; in fact, their work is based on the same inequality$^4$ that we use to prove (ii*).

6. A convexity theorem for the variation-norm. With (3) as a starting-point, it is easy to define an extension $F$ of the function

$^3$ Due to L. C. Young [8]. The articles by Love and Young involve only scalar-valued functions.
\{(g, r) \mapsto V_0(E), \} \text{ such that } \log F(\alpha^{-1}, \beta^{-1}) \text{ is a convex function of } (\alpha, \beta) \text{ in the rectangle } 0 \leq \alpha, \beta \leq 1. \text{ This fact plays a basic role in proving the results that have been presented.}

7. Remarks. For more details concerning the Banach space \(W_p(J)\), see [5]. If \(p = 1\), then \(V_p(f)\) is the total variation of \(f\). The class \(W_p(J)\) becomes a Banach algebra under pointwise multiplication (and under the norm \(\{f \mapsto \|f\|_1 + V_p(f)\}\)). The article [5] deals with continuous linear functionals on \(W_p(J)\).

REFERENCES


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