ON H. WEYL'S CHARACTER FORMULA

BY P. CARTIER

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Introduction. Many years ago, H. Weyl [**4**] gave a general formula for the characters of a compact Lie group, or what amounts to the same thing, of a complex semi-simple Lie group. His proof leaned on a fundamental integration formula and was analytical and topological. Later on an algebraic proof was supplied by H. Freudenthal [**2**]. Quite recently, B. Kostant [**3**] gave a rather explicit formula for the multiplicity of a weight μ in an irreducible representation with maximal weight λ. The purpose of the present note is a proof for the equivalence of Weyl's and Kostant's formulae; since our proof is very simple, the benefit of Kostant's paper is a new algebraic proof for Weyl's formula. It goes without saying that [**3**] is of considerable independent interest for the other results it contains.

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1. **Notations.** Let G be a complex Lie group, $\mathfrak{g}$ its Lie algebra, B the Killing bilinear form on $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Σ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$; therefore Σ is a set of linear forms on $\mathfrak{h}$. For each root α there exists a unique element H_α in $\mathfrak{h}$ such that $\alpha(H_\alpha)=2$ and the linear form $H\to B(H, H_\alpha)$ on $\mathfrak{h}$ be proportional to α. The symmetry S_α associated to the root α is the linear automorphism of $\mathfrak{h}$ given by $S_\alpha(H)=H-\alpha(H)\cdot H_\alpha$; the group W generated by the S_α's is called the Weyl group; it is finite and Σ is stable under W.

Let us choose now a fundamental set of roots $\Pi=\{\alpha_1, \cdots, \alpha_l\}$; that means let Π be a set of roots and any root be a linear combination of the roots in Π with integral coefficients all of the same sign; this common sign is called the sign of the root (with respect to Π). By ϕ we mean the half-sum of all positive roots.

Let now π be any irreducible representation of $\mathfrak{g}$ in a complex vector space V. For any linear form μ on $\mathfrak{h}$ let V_μ be the set of all v in V such that $\pi(H)\cdot v=\mu(H)\cdot v$ for any H in $\mathfrak{h}$; then μ is a weight of π means $V_\mu\neq 0$ and the multiplicity of μ is the dimension of V_μ. As is well known, there exists a weight λ of multiplicity 1, such that any other weight is of the form $\lambda-\sum_{1\leq i\leq l} m_i\cdot\alpha_i$ with integers $m_i\geqq 0$ not all zero. The representation π is defined up to equivalence by λ and we write $\pi=\pi_\lambda$ to mean that λ is the "maximal" weight of π. Those linear forms λ on $\mathfrak{h}$ are candidates for a maximal weight for which $\lambda(H_\alpha)$ is an integer $\geqq 0$ for any α in Π.

2. **Character formula.** With the previous notations, Weyl's formula is as follows:

$$\mathrm{Tr}(\pi_\lambda(\exp H)) = \frac{\sum_{s\in W} \det s\cdot e^{(\phi+\lambda)(s\cdot H)}}{\sum_{s\in W} \det s\cdot e^{\phi(s\cdot H)}}. \tag{1}$$

We explain that H is any element in $\mathfrak{h}$, that exp means the exponential mapping from $\mathfrak{g}$ to G as defined by Chevalley [1], and $\mathrm{Tr}(A)$ is the trace of an operator A on V. According to Weyl, the denominator in (1) can be rewritten in the following form:

$$\sum_{s\in W} \det s\cdot e^{\phi(s\cdot H)} = \prod_{\alpha \text{ positive root}} (e^{+\alpha(H)/2} - e^{-\alpha(H)/2}). \tag{2}$$

Let us now give Kostant's formula. For any linear form μ on $\mathfrak{h}$, the dimension of V_μ is denoted $m_\lambda(\mu)$ if λ is the maximal weight of the representation π of $\mathfrak{g}$ in V. Let $P(\mu)$ be the "number of partitions of μ into positive roots," that is, precisely the number of all functions $\alpha\to n_\alpha$ defined for positive roots α and with positive integral values such that $\mu=\sum_\alpha n_\alpha\cdot\alpha$; $P(\mu)$ is the coefficient of $e^{-\mu}$ in the Fourier development for the product $\prod_{\alpha \text{ positive root}} 1/(1-e^{-\alpha})$. According to Kostant we get

$$m_\lambda(\mu) = \sum_{s\in W} \det s\cdot P(s(\phi+\lambda) - (\phi+\mu)). \tag{3}$$

3. **Proof of equivalence.** For any H in $\mathfrak{h}$ the operator $A=\pi_\lambda(\exp H)$ is diagonalizable; more precisely, on V_μ it induces the dilatation of ratio $e^{\mu(H)}$. Its trace is therefore equal to $\sum_\mu m_\lambda(\mu)\cdot e^{\mu(H)}$; this means $m_\lambda(\mu)$ is the coefficient of e^μ in the Fourier development for the left side of (1). Furthermore the function of H given by (2) is equal to

$$e^{\phi(H)}\cdot \prod_{\alpha \text{ positive root}} (1 - e^{-\alpha(H)})$$

and by definition of $P(\mu)$ its inverse is given by $\sum_\nu' P(\nu)\cdot e^{-(\phi+\nu)(H)}$. For the right side of (1) we get

$$\sum_\nu \sum_{s\in W} \det s\cdot e^{(\phi+\lambda)(s\cdot H)-(\phi+\nu)(H)}\cdot P(\nu)$$
$$= \sum_\mu \sum_{s\in W} \det s\cdot P(s^{-1}(\phi+\lambda) - (\phi+\mu))\cdot e^{\mu(H)}$$

since by definition $\rho(s\cdot H)=(s^{-1}\rho)(H)$ for any s in W, H in $\mathfrak{h}$ and any linear form ρ on $\mathfrak{h}$. Since $\det s=\pm 1$ for any s in W, we get $\det s=\det s^{-1}$; therefore the right member of (3) is the coefficient of e^{μ} in the Fourier development for the right side of (1).

This finishes the proof, which looks definitely shorter than the preliminary explanations!

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CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE, PARIS, FRANCE