BOOK REVIEWS

Éléments de géométrie algébrique. Par A. Grothendieck, rédigés avec la collaboration de J. Dieudonné. Publications de l'Institut des Hautes Études Scientifiques No. 4, Paris, 1960. 228 pp. 27 NF.

The present work, of which Chapters 0 and I are now appearing together, is one of the major landmarks in the development of algebraic geometry. It plans to cover eventually everything that is known in algebraic geometry over arbitrary ground rings, and of course a lot more besides. A tentative list of its chapters is as follows:

Chapter I. Le langage des schémas.
II. Étude globale élémentaire de quelques classes de morphismes.
III. Cohomologie des faisceaux algébriques cohérents. Applications.
IV. Étude locale des morphismes.
V. Procédés élémentaires de construction de schémas.
VI. Technique de descente. Méthode générale de construction de schémas.
VII. Schémas de groupes, espaces fibrés principaux.
VIII. Étude differentielle des espaces fibrés.
IX. Le groupe fondamental.
X. Résidus et dualité.
XI. Théories d'intersection, classes de Chern, théorème de Riemann-Roch.
XII. Schémas abeliens et schémas de Picard.
XIII. Cohomologie de Weil.

The list is subject to modifications, especially in so far as later chapters are concerned, partly because much of the research needed to complete these chapters remains to be done.

To give the prospective reader some idea of the size of the work, suffice it to say that Chapter I is 134 pages long, that subsequent chapters are expected to be at least as long (probably around 150 pages each), that all chapters are regarded as being open (i.e., subject to additions such as are deemed necessary in the course of the writing), and that Chapters 0 and I together weigh 1 and 3/4 pounds in their present form.

In order to get a more specific idea of what is to come, one should consult first Grothendieck's address to the International Congress at

Edinburgh, 1958, and also the whole series of talks at Bourbaki seminars given in the past two years (available at the Institut Henri Poincaré, 11 Rue Pierre Curie, Paris) in which he has given a sketch of the proofs of important results to appear in later chapters. These talks will provide the necessary motivation to the whole work. They are written concisely, directly, and excitingly. Such motivation could not be given in the actual text, which is written very lucidly, is perfectly organized, and very precise. Thanks are due here to Dieudonné, without whose collaboration the labor involved in writing and publishing the work would have been insurmountable.

Before we go into a closer description of the contents of Chapters 0 and I, it is necessary to say a few words explaining why the present treatise differs radically in its point of view from previous ones.

1. Most of algebraic geometry up to now has been concerned with varieties, say over arbitrary fields. It includes some results on algebraic families of varieties, but such results are few in number, and it has become increasingly clear in recent years that one was facing serious difficulties in dealing with such algebraic systems. For example, the geometer is able to attach to a fixed variety other geometric objects, say a Picard variety. It is then a problem to show that if one has an algebraic system of varieties, the Picard varieties can be associated in such a way that they move along with the varieties, following the same parameter variety, even when special members of the family are degenerate. The tools available at present to deal with such a problem are recognized to be deficient (although of course in special cases, interesting results have been obtained, especially for non-degenerate fibers).

2. In applications to number theory, it has been realized for some time that the reduction mod $\mathfrak{p}$ of a variety defined over a number field was completely analogous to the situation of algebraic systems, a fiber being such a reduction. Although it was possible here again to give an ad hoc definition and results having useful applications to interesting special problems, the theory was technically disagreeable to apply, to say the least.

In order to deal efficiently with the above two points, it was necessary to incorporate from the start into the foundations the notion of a variety defined over a ring, not necessarily Noetherian, and having nilpotent elements (say to reduce mod $\mathfrak{p}^n$, or to describe degenerate fibers in a system). This meant that a variety could not be regarded any more as a model of a "function field," and thus that it should be defined starting with a local description supplemented by a method for gluing local pieces together (sheaves being the natural tool here).

3. The classical tools available were impotent to deal with the problem of defining the homology and homotopy functors to which one is accustomed in topology, and having similar properties. The necessity of having the homology functor, say, was made clear by Weil, who pointed out that if one has it, then the structure of the zeta function for non-singular projective varieties defined over finite fields follows immediately from the Lefschetz fixed point formula. In order to have this, a minimum requirement is that the homology groups H_n associated with a variety V be modules, or vector spaces having characteristic 0 (no matter what the characteristic of the field of definition of V is!).

4. The study and classification of non-abelian coverings of varieties, and in particular the determination of the fundamental group, was completely outside the range of available methods, except for varieties defined over the complex numbers where one could use transcendental methods.

The above list could be expanded, but it gives a good idea why a new approach to algebraic geometry was needed.

Let us now give a closer look at the contents of Chapters 0 and I.

Chapter 0 is intended to include results of commutative algebra needed for the geometric applications. They are more or less well known, but it is difficult to give references for them. The reader should skip this chapter until he meets a place where he needs it. He should start reading Chapter I immediately. For this, he needs to know only what a ring is (commutativity and unit element are always assumed), and the definition of a ring of fractions, which runs as follows. Let A be a ring, S a subset of A closed under multiplication and containing 1. One considers equivalence classes of pairs (a, s) with $a \in A$ and $s \in S$ such that $(a, s) \sim (a', s')$ if there exists $s_1 \in S$ such that $s_1(s'a - sa') = 0$. The equivalence class of (a, s) is denoted by a/s, and these form a ring in the obvious way. This ring is denoted by $S^{-1}A$, and is 0 if S contains nilpotent elements. The most important case is that where S is the complement of a prime ideal $\mathfrak{p}$, so that $S^{-1}A = A_{\mathfrak{p}}$ is the local ring at $\mathfrak{p}$.

We recall that a ringed space is a pair (X, O_X) consisting of a topological space X and a sheaf of rings O_X. Ringed spaces form a category: A morphism $(X, O_X) \to (Y, O_Y)$ is a pair consisting of a continuous map $\phi: X \to Y$ and a contravariant map $\psi: O_Y \to O_X$ compatible with f. If we denote by O_x the fiber of O_X above a point $x \in X$, then ψ induces a homomorphism $\psi_x: O_{\phi(x)} \to O_x$. The ringed space (X, O_X) is called a local ringed space if all the rings O_x are local rings. If (X, O_X) and (Y, O_Y) are local ringed spaces, a morphism (ϕ, ψ) above is called local if the inverse image of the maximal ideal of O_x by ψ_x is the maximal ideal of $O_{\phi(x)}$. The local ringed spaces and the local morphisms then form a category. It is a subcategory of this one which is of interest to the algebraic geometer.

Namely, given a ring A, its spectrum $X = \operatorname{spec}(A)$ is the topological space (T_0 but not T_1) whose points are the prime ideals of A with Zariski topology (the set of primes containing a given ideal is closed). One views X as a ringed space, the sheaf being that of the local rings $A_{\mathfrak{p}}$. It is thus a local ringed space, called an affine scheme. A prescheme is a local ringed space (X, O_X) such that every point admits an open neighborhood U such that $(U, O_X|U)$ is isomorphic to an affine scheme. The preschemes form a category, the morphisms being the local morphisms.

To simplify the notation, one sometimes omits the structure sheaf O_X and the map ψ, just writing for instance $\phi: X \to Y$ to indicate a morphism in the category of preschemes.

Let Γ be the functor "section". For each open subset U of X, ΓU is the ring of sections of O_X over U. Given a morphism $\phi: X \to Y$, we have a homomorphism $\Gamma(\phi): \Gamma Y \to \Gamma X$. The converse is true for affine schemes, and in fact affine schemes Y are characterized among preschemes by the fact that for each prescheme X the map $\phi \to \Gamma(\phi)$ of $\operatorname{Mor}(X, Y)$ into $\operatorname{Hom}(\Gamma Y, \Gamma X)$ is an isomorphism. (One could actually let X range over local ringed spaces.) Furthermore, if $Y = \operatorname{spec}(A)$, then ΓY is naturally isomorphic to A.

The other main result of Chapter I is then given: It is the proof that products exist in the category of preschemes. Let us recall some terminology in abstract categories. Let C be a category, and S an object in C. We denote by C_S the category of objects over S, i.e. pairs (X, f) where X is in C and f is a morphism $f: X \to S$ in C, called the structural morphism. Given two objects $f: X \to S$ and $g: Y \to S$ in C_S, a morphism ϕ in C_S is a morphism $\phi: X \to Y$ in C which is such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

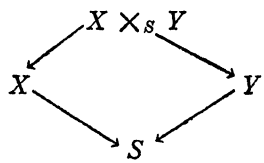
is commutative.

In the category of preschemes, the object S plays the role of a ground object (ground field, ground ring, ground anything you want vastly generalized, parameter object, etc.).

A product of two objects (X, f) and (Y, g) over S consists of an object (written $X\times_S Y$) and two morphisms

$$\phi: X \times_S Y \to X$$
$$\psi: X \times_S Y \to Y$$

making the following diagram commutative and satisfying the obvious universal mapping property for such pairs of maps:



It is uniquely determined, up to a unique isomorphism.

If A, B, R are three rings, and A, B are algebras over R, then the product of the two affine schemes spec(A) and spec(B) over spec(R) is spec$(A \otimes_R B)$, the morphisms involved being the obvious ones. This is practically immediate from the definitions, and the existence proof in the general case is carried out by gluing local pieces together.

One can consider the product non-symmetrically. Viewing S as ground object, let $S' = Y$ be viewed as an extension of it. Then $X\times_S S'$ (sometimes written $X^{S'}$) may be viewed as an object over S', called the pull back of X by the morphism $g: S' \to S$. This pull back involves as a special case the extension of ground field or ring, and also reduction mod p, or the process of taking a fiber. For instance, if $S = \text{spec}(Z)$ (Z the integers), then for each prime p, we have a morphism

$$\text{spec}(Z/pZ) \to \text{spec}(Z)$$

and thus for each prescheme X over Z, we get its fiber over Z/pZ, namely $X\times_Z \text{spec}(Z/pZ)$.

Having constructed products, one gets a diagonal morphism

$$X \to X \times X$$

(the product without subscript being always over spec(Z)). One says that X is a scheme if this morphism is closed (obvious definition).

Most of the rest of Chapter I is devoted to defining certain classes of morphisms in the category of preschemes (immersions, closed immersions, local immersions, morphisms of finite type, proper morphisms, separated morphisms, etc. and in subsequent chapters affine morphisms, projective morphisms, flat morphisms, unramified morphisms, simple morphisms, ad lib.) and of proving standard properties concerning the composition and products of such special classes of morphisms. Namely, given a category C, let us say that a subclass C' of morphisms of C is distinguished if it has the following properties:

(i) If f, g are in C' and can be composed, so is fg.

(ii) If $f: X \to S$ is in C' and $g: Y \to S$ is in C, then the pull back of f by g is in C'.

(iii) If both f and g are in C', so is $f \times_S g$.

(iv) If f and g can be composed, g is in C' and gf is in C', then f is in C'.

The general rule is that all particular types of morphisms defined in Chapter I (and subsequently) will form a distinguished subclass, except possibly under certain conditions of finiteness and separation. There is no point in going into the specific details here. We wish merely to indicate the way the system works.

Chapter I concludes with an extended discussion of quasi-coherent sheaves, and formal schemes, those arising essentially from completions of topological rings, and playing an important role in local analytic (algebraic) questions. They are not used until Chapter III, which will include Zariski's theory of holomorphic functions and the connectedness theorem, and the reader may skip that part until he needs it.

One more notion appears in Chapter I, worthy of notice for the implications it has concerning the point of view of the work. Again it is best to describe it in an abstract category C. Let A be a fixed object in C and let X vary in C. Then

$$F_A: X \to \mathrm{Mor}(X, A)$$

is a (contravariant) functor from C into the category of sets, denoted Ens. We may also denote $\mathrm{Mor}(X, A)$ by $A(X)$ and in our category of preschemes, we think of it as giving the set of points of A in X. (To justify this, think of A as an affine variety V over a field k, and let Γ be its finitely generated algebra of functions over k. Let K range over fields containing k. Then points of V in K are in bijective correspondence with homomorphism of Γ into K, i.e. morphisms of $\mathrm{spec}(K)$ into $\mathrm{spec}(\Gamma)$. Here, $\mathrm{spec}(K)$ consists of one point, and the local ring above it is just K itself.)

Given a functor $F: C \to \mathrm{Ens}$ of C into the category of sets, Grothendieck calls F representable if it is isomorphic to a functor of type F_A. (The functors of one category into another form themselves a category, the morphisms being the obvious ones.) It is then immediate that the object A is uniquely determined, up to a unique isomorphism.

Observe that the definition of products has been made in accordance with the representation functor, i.e. to satisfy the formula

$$(X \times_S Y)(T) \approx X(T) \times_{S(T)} Y(T)$$

for all objects T, the fiber product on the right being the usual one in the theory of sets (pairs of points projecting on the same point in $S(T)$).

This notion of representable functor allows one to transport to any category standard notions like group, ring, etc. For instance, an object G is called a group object if one is given two morphisms $G \times G \to G$ (composition) and $G \to G$ (inverse) such that the representation functor into the category of sets defines a group structure on the set $G(X)$ for each X. (We have assumed finite products exist, but a rephrasing would do away with this.)

It is one of the most basic ones of mathematics. To give an example from topology: On the category of CW complexes, the functor H^n_π is representable by $K(\pi, n)$. Or on the category of reasonable topological spaces, the functor K (classes of vector bundles) is also representable by the classifying space.

In algebraic geometry, Grothendieck reformulates certain classical problems in terms of the representation of functors, for instance the problem of constructing Picard schemes. Given X over S, the Picard functor consists in associating to each T over S the divisor classes of X which are rational over T. (This can of course be made precise.) The Picard scheme, if it exists, represents this functor. Grothendieck has recently obtained a fairly general condition on functors in the category of schemes under which he can prove that a functor is representable. This point of view marks a complete discontinuity with those preceding it and in a certain sense, is the first essentially new approach having entered algebraic geometry since the Italian school.

A theorem is not true any more because one can draw a picture, it is true because it is functorial.

To conclude this review, I must make a remark intended to emphasize a point which might otherwise lead to misunderstanding. Some may ask: If Algebraic Geometry really consists of (at least) 13 Chapters, 2,000 pages, all of commutative algebra, then why not just give up?

The answer is obvious. On the one hand, to deal with special topics which may be of particular interest only portions of the whole work are necessary, and shortcuts can be taken to arrive faster to specific goals. Thus one may expect a period of coexistence between Weil's

Foundations and *Elements*. Only history will tell if one buries the other. Projective methods, which have for some geometers a particular attraction of their own, and which are of primary importance in some aspects of geometry, for instance the theory of heights, are of necessity relegated to the background in the local viewpoint of *Elements*, but again may be taken as starting point given a prejudicial approach to certain questions.

But even more important, theorems and conjectures still get discovered and tested on special examples, for instance elliptic curves or cubic forms over the rational numbers. And to handle these, the mathematician needs no great machinery, just elbow grease and imagination to uncover their secrets. Thus as in the past, there is enough stuff lying around to fit everyone's taste. Those whose taste allows them to swallow the *Elements*, however, will be richly rewarded.

S. LANG

Foundations of Modern Analysis. By J. Dieudonné. New York, Academic Press, 1960. 14+361 pp. $8.50.

The purpose of this book is to provide the necessary elementary background for all branches of modern mathematics involving Analysis, and to train the students in the use of the axiomatic method. It emphasizes conceptual rather than computational aspects. Besides pointing out the economy of thought and notation which results from a general treatment, the author expresses his opinion that the students of today must, as soon as possible, get a thorough training in this abstract and axiomatic way of thinking if they are ever to understand what is currently going on in mathematical research. The students should build up this "intuition of the abstract", which is so essential in the mind of a modern mathematician. The angle from which the content of this volume is considered is different from the ones in traditional texts of the same level because the author does not just imitate the spirit of his predecessors but instead has a more independent pedagogical attitude. This book takes the students on a tour of some basic results, among them the Tietze-Urysohn extension theorem, the Stone-Weierstrass approximation theorem, the Ascoli compactness theorem, the Jordan curve theorem and the F. Riesz perturbation theory. These are some of the hills in the scenery which are surrounded by nice valleys connecting them. This course, to be taught during a single academic year, is *elementary* in the sense that it is intended for *first year* graduate students or exceptionally advanced undergraduates. Naturally, students must have a good work-