RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A NEW CLASS OF PROBABILITY LIMIT THEOREMS

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Suppose that $\{X_n\}$ is a Markov process with states on the non-negative real axis and stationary transition probabilities. Define

$$\mu_k(x) = E[(X_{n+1} - X_n)^k \mid X_n = x], \qquad k = 1, 2, \cdots; \tag{1}$$

we assume that for each k, $\mu_k(x)$ is a bounded function of x. Assume also

$$\lim_{x\to\infty} \mu_2(x) = \beta > 0, \qquad \lim_{x\to\infty} x\mu_1(x) = \alpha > -\frac{\beta}{2}\cdot \tag{2}$$

We shall say that the process $\{X_n\}$ is *null* provided that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Pr(X_i \leqq M) = 0 \tag{3}$$

for all finite M. A class of examples satisfying all the conditions imposed so far is afforded by Markov chains on the integers with transition probabilities of the form

$$p_{j,j+1} = \frac{1}{2}\left[1 + \frac{\alpha}{j} + o\left(\frac{1}{j}\right)\right] > 0,\ p_{j,j-1} = 1 - p_{j,j+1} \quad \text{if } j \neq 0; \tag{4}$$

$$p_{01} = 1 - p_{00} > 0.$$

For such chains (random walks) the *null* condition is known to hold if $\alpha > -1/2$ $(= -\beta/2)$. In many (but, so far at least, not all) other cases, it can be shown that (3) follows automatically from the other hypotheses.

For a process $\{X_n\}$ satisfying the above assumptions, there is an analogue of the central-limit theorem:

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THEOREM 1.

$$\lim_{n\to\infty} \Pr(X_n \leqq y(n)^{1/2}) = \int_0^y \frac{2\xi^{2\alpha/\beta}e^{-\xi^2/2\beta}}{(2\beta)^{\alpha/\beta+1/2}\Gamma\left(\frac{\alpha}{\beta}+\frac{1}{2}\right)}\, d\xi. \tag{5}$$

This seems to be a novel result even for random walks (despite the extensive recent development of their theory), and was reported in [3]. Under very slightly stronger hypotheses, however, much more is true. We shall call the process $\{X_n\}$ *uniformly null* provided the limit (3) holds uniformly in the initial state X_0. Again it can be shown that this often follows automatically; in particular, it holds for the random walks (4). For such processes we can prove

THEOREM 2. *For any* $t>0$,

$$\lim_{n\to\infty} \Pr(X_{[nt]} \leqq y(n)^{1/2} \mid X_0 = x(n)^{1/2}) = p_t(x, y) \tag{6}$$

exists; the limit $p_t(x, y)$ *is the transition-probability function for the diffusion process with backward equation*

$$u_t = \frac{\alpha}{x} u_x + \frac{\beta}{2} u_{xx} \qquad (\alpha \text{ and } \beta \text{ are as in } (2)) \tag{7}$$

and with a reflecting barrier (if necessary) at the origin.

With the aid of these results it is easy to see that there is an analogue of the multi-dimensional C.L.T.; that is, the limit of

$$\Pr(X_{[nt_1]} \leqq y_1(n)^{1/2}, \cdots, X_{[nt_k]} \leqq y_k(n)^{1/2})$$

can be calculated. It is then natural to seek the appropriate version of the Erdös-Kac-Donsker *invariance principle* [1]. Define a continuous function $x_t^{(n)}$ by setting

$$x_t^{(n)} = \frac{X_i}{n^{1/2}} \quad \text{when} \quad t = \frac{i}{n}, \qquad i \leqq n, \tag{8}$$

and by linear interpolation for other t. Let C be the space of all continuous functions x_t on $[0, 1]$ with $x_0=0$, and endow C with the uniform topology. Our main result is

THEOREM 3. *Under the conditions of Theorem* 2,

$$\lim_{n\to\infty} \Pr(f(x_t^{(n)}) \leqq \alpha) = \Pr(f(x_t) \leqq \alpha), \tag{9}$$

where x_t is the diffusion process encountered in Theorem 2, and where $f(\ )$ is a functional on C continuous almost everywhere with respect to the measure of the process $\{x_t\}$.[2]

From this a large number of interesting limit theorems follow (as from Donsker's theorem) by choosing specific functionals $f(\cdot)$. An important example for which the limit distribution can be obtained more or less explicitly is the case $f(x_t) = \max\{x_t | 0 \leqq t \leqq 1\}$.

Theorems 1 and 2 are reminiscent of a general limit theorem in Khintchine [2], and Theorem 3 of recent work of Prokhorov [4] and Skorohod [5]. None of these general results seem to be directly useful in proving the above theorems, however. Our proofs, together with additional results and applications, some extensions, and more complete references, will be published separately in the near future. It might be remarked that the methods are, for the most part, quite elementary. Calculations with moments and use of the moment-convergence theorem are prominent in the proofs of Theorems 1 and 2, while that of Theorem 3 is analogous in large measure to Donsker's procedure in [1].

References

1. M. Donsker, *An invariance principle for certain probability limit theorems*, Memoirs Amer. Math. Soc. no. 6, 1951.

2. A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, New York, Chelsea, 1948 (reprinted).

3. J. Lamperti, *Limit theorems for certain stochastic processes*, Abstract 569-32 Notices Amer. Math. Soc. vol. 7 (1960) pp. 268–269.

4. Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i Primenen vol. 1 (1956) pp. 289–319.

5. A. V. Skorohod, *Limit theorems for Markov processes*, Teor. Veroyatnost. i Primenen vol. 3 (1958) pp. 217–264.

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[2] It is understood that the diffusion process satisfies the initial condition $x_0 = 0$ and that the convergence in (9) is for all α for which the right-hand side is continuous.