RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ONTO INNER DERIVATIONS IN DIVISION RINGS

BY E. E. LAZERSON

Communicated by N. Jacobson, March 17, 1961

1. Introduction. Kaplansky [3] proposed the following problem: Does there exist a division ring $\Delta$ each element of which is a sum of additive commutators $ab - ba$? In [1] Harris gave a strongly affirmative solution to this problem by constructing division rings $\Delta$ in which each element $c = ab - ba$ for some $a, b \in \Delta$. Recently Meisters [4] has studied rings $R(0)$ in which for any triple of elements $a, b, c \in R$ with $a \neq b$ there exist solutions of the equation $ax - xb = c$. He has shown that (1) $R$ is a division ring in which every noncentral element induces an onto inner derivation and (2) if $R$ is separable algebraic over its center, then $R$ is commutative. Actually one can prove the more general result that in a division ring $R$ of the preceding type all algebraic elements (over the center) are central. (Hence if $R$ is noncommutative, each noncentral element $t \in R$ is transcendental over the center of $R$ and induces an onto inner derivation.)

In view of the above work it seems natural to investigate the question of existence of division rings possessing onto inner derivations. We give a partial answer to this question which implies (in some heuristic sense) that Harris' examples (at least for char. $p > 0$) are normative rather than pathological. More precisely we sketch a proof of the following theorem: For each division ring $\Delta$ of char. $p > 0$ one can construct an extension division ring $E$ with the property that there exists an element $t \in E$ (lying in the centralizer of $\Delta$) whose associated inner derivation $D_t$ is an onto map: $D_t(E) = E$.

2. Preliminaries. We shall make consistent use of the following facts: (1) Any noncommutative ring $R$ with an identity having the common right multiple property has a right quotient ring $Q(R)$, i.e., every element of $Q(R)$ has the form $ab^{-1}, a, b \in R$, $b$ regular, and all regular elements of $R$ are invertible in $Q(R)$. (2) If $\Delta$ is a division ring and $D$ a derivation of $\Delta$ into itself, then $\Delta[x; D]$, the ring of differential polynomials over $\Delta$ in the indeterminate $x$, has the com-
mon right multiple property; thus by (1), $\Delta[x; D]$ has a quotient division ring $Q(\Delta[x; D])$, since all nonzero elements in $\Delta[x; D]$ are regular. (3) If $R$ is a ring with quotient ring $Q(R)$ and $D$ is a derivation of $R$ into an extension ring $S$ of $Q(R)$, then $D$ can be uniquely extended to a derivation of $Q(R)$ into $S$ by defining, for $ab^{-1} \in Q(R)$, $D(ab^{-1}) = D(a)b^{-1} - (ab^{-1})(D(b)b^{-1})$.

A proof of (1) may be found in [2, p. 118]; (2) was established in [5]; and (3) is a fairly straightforward exercise in computation. Finally note that in rings of char. $p > 0$ all $p^n$th powers ($n \geq 0$) of a derivation are again derivations.

3. The construction. Let $\Delta_0$ be the quotient division ring of the polynomial ring $\Delta[t]$ ($\Delta$ a division ring of char. $p > 0$) where $t$ is a commuting indeterminate over $\Delta$. Set $x_0 = 1$ and let $D_0$ be the unique extension of ordinary differentiation in $\Delta[t]$ to $\Delta_0$ so that $D_0$ is a derivation of $\Delta_0$ into itself. Choose an indeterminate $x_1$ over $\Delta_0$ and form the quotient division ring $\Delta_1 = Q(\Delta_0[x_1; D_0])$. Noting that $D_1(x_1) = x_0$ and $D_0(x_0) = 0$, we see that we have verified the case $n = 0$ of the proposition: Given $\Delta_0 = Q(\Delta[t])$ there exists a nested sequence of division rings $\Delta_n$, a set of derivations $D_n: \Delta_n \rightarrow \Delta_n$, and elements $x_n \in \Delta_n$ satisfying

\begin{align*}
(1) & \quad \Delta_{n+1} = Q(\Delta_n[x_{n+1}; D_n]), \\
(2) & \quad D_t(x_{n+1}) = x_n, \\
(3) & \quad D_n(t) = x_n, \quad D_n(x_i) = 0, \quad i = 0, \ldots, n; \quad n \geq 0.
\end{align*}

To prove this proposition we proceed by induction. Suppose the truth of the proposition for $n = 0, \ldots, s$. Then we have constructed $\Delta_n$, $D_n$, $x_n$ for $n = 0, \ldots, s$, satisfying the above conditions. Choose an indeterminate $x_{s+1}$ over $\Delta_s$ and let $\Delta_{s+1} = Q(\Delta_s[x_{s+1}; D_s])$. We must construct a derivation $D_{s+1}: \Delta_{s+1} \rightarrow \Delta_{s+1}$ satisfying $D_{s+1}(t) = x_{s+1}$, $D_{s+1}(x_i) = 0$ ($i = 0, \ldots, s+1$), and $D_t(x_{s+1}) = x_s$. We do this by defining $D_{s+1}$ on $\Delta_0$ and extending it to each successive $\Delta_i$ ($i = 1, \ldots, s+1$) as follows. Suppose $D_{s+1}$ has been defined on $\Delta_i$, $0 \leq i < s+1$; then to define it on $\Delta_{i+1}$ we need only check that it can be extended to $\Delta_{i}[x_{i+1}; D_t]$. Now if $\sum a_ix_{i+1}^t$, $a_i \in \Delta_i$, is a typical element of this ring we set $D_{s+1}(\sum a_ix_{i+1}^t) = \sum D_{s+1}(a_i)x_{i+1}^t$. Since the map $D_{s+1}D_t - D_tD_{s+1}$ is zero on $\Delta_t$, one verifies that $D_{s+1}$ as defined is a derivation on $\Delta_{i+1}$. Thus if $D_{s+1}$ can be constructed on $\Delta_0$ we shall be done. Let $a \in \Delta[t]$. Define $D_{s+1}(a) = \sum_{i=0}^{s+1} D_0^{i+1}(a)/(i + 1)!x_{i+1-i}$ (mod $p$).
This makes sense since the coefficients of $D_{i+1}^l(a)$ are divisible by $(i+1)!$. Observing that $x_i a = \sum_{l=0}^i D_l^0 a / i! x_{l-1} \pmod{p}$, $l = 0, \cdots, s+1$, one verifies that $D_{i+1}$ is a derivation on $\Delta[t]$ and hence on $\Delta_0$. By what we have said previously it has an extension to $\Delta_{s+1}$ and clearly satisfies all requisite properties.

Next let $E = \bigcup_{n=0}^s \Delta_n$. Since $D_i(x_n) = x_{n-1}$ we get $D_{i+1}^n(x_n) = 0$ and therefore there exists a least integer $l \geq 0$ for which $D_{i+1}^l(x_n) = 0$. It is immediate that $D_{i+1}^l(\Delta_n) = 0$, so $\Delta_n$ is contained in the centralizer of $t^p$. But $D_{i+1}^l(x_{p^i}) = 1$, hence if $a$ is in the centralizer of $t^p$: $x_{p^i} a t^{p^i} - t^{p^i} x_{p^i} a = a$. It follows, since $x_{p^i} a$ is in $\Delta_{p^{i+1}}$, that $D_{i+1}^l(\Delta_{p^{i+1}}) \supseteq \Delta_n$. But $D_i(\Delta_{p^{i+1}}) \supseteq D_{i+1}(\Delta_{p^{i+1}}) \supseteq \Delta_n$. As $n$ was arbitrary, $D_i(E) = E$.

**REFERENCES**


**Institute for Defense Analyses**