CORRECTION TO “SPACES OF RIEMANN SURFACES AS BOUNDED DOMAINS”

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In [3] I sketched a proof of the theorem: Every Teichmüller space $T_{g,n}$ is a bounded domain in complex number space.

This proof is invalid since Lemma B is false. The error in the argument occurs on page 101, lines 8–12. The theorem is nonetheless true. A complete proof will appear elsewhere; a brief outline follows. The same proof was found, simultaneously and independently, by Lars V. Ahlfors.

Let $G$ be a Fuchsian group without elliptic elements and with the unit circle as a limit circle. Denote by $U$ the unit disc and by $V$ the domain $1 < |z| < \infty$. The Riemann surfaces $S = V/G$ and $\overline{S} = U/G$ are mirror images of each other.

Let $M$ be the set of complex-valued measurable functions $\mu(z)$ such that $|\mu(z)| \leq k(\mu) < 1$, $\mu = 0$ in $U$, and $\mu(z)\overline{dz}/dz$ is invariant under $G$. For $\mu \in M$ let $z \mapsto w^\mu(z)$ be the homeomorphism of the plane onto itself which satisfies the Beltrami equation $w_\mu = \mu w_z$ and is normalized by the conditions $w^\mu(0) = 0$, $w^\mu(1) = 1$. Then $G^\mu = w^\mu G(w^\mu)^{-1}$ is a discontinuous group of Möbius transformations and $S^\mu = w^\mu(V)/G^\mu$ a Riemann surface. Also, $w^\mu$ defines a quasiconformal mapping $f^\mu$ of $S$ onto $S^\mu$ and thus a point in the Teichmüller space $T(S)$; all points in this space can be so obtained. We say that $\mu$ and $\nu$ are equivalent if they define the same point in $T(S)$, i.e. if $S^\mu$ is conformal to $S^\nu$ and $f^\mu$ homotopic to $f^\nu$. This is so if and only if there is a Möbius transformation $C$ such that $C(w^\mu(z)) = w^\nu(z)$ in $U$ (cf. [2]).

Holomorphic quadratic differentials on $\overline{S}$ may be represented by $G$–automorphic forms of weight $(-4)$ in $U$, i.e. by holomorphic functions $\phi(z)$, $z \in U$, with $\phi(z)dz^2$ invariant under $G$. We define the norm $||\phi||$ to be the supremum of $\lambda |\phi|$ where $\lambda(z) = (1 - |z|^2)^2$. The quadratic differentials of finite norm form a complex Banach space $B$.

For $\mu \in M$ the function $w^\mu$ is holomorphic in $U$ and so is its Schwarz derivative $\phi^\mu$; note that $\phi^\mu$ depends only on the equivalence class $[\mu]$ of $\mu$. One verifies directly that $\phi^\mu(z)dz^2$ is $G$–invariant, and by a theorem of Nehari [4] we have that $||\phi^\mu|| \leq 6$. Knowing $\phi^\mu$ we may recon-
struct $w^\mu$ as the quotient of two solutions of the ordinary differential equation $2\kappa'' = \phi \kappa$. Hence $[\mu] \rightarrow \phi^\mu$ is a one-to-one mapping of $T(S)$ onto a bounded set $W \subset B$. This mapping is holomorphic in the following sense: if $\mu \in M$ depends holomorphically on complex parameters, so does $\phi^\mu(z)$, for every fixed $z \in U$ (cf. [1]).

Assume now that $G$ is finitely generated. Then $S$ is obtained from a closed surface of genus $g$ by removing $n \geq 0$ points with $3g - 3 + n > 0$, $\dim B = 3g - 3 + n$, and $[\mu] \rightarrow \phi^\mu$ is a holomorphic homeomorphism of $T(S) = T_{g,n}$ onto $W$. Since $\dim T_{g,n} = \dim B$, $W$ is a domain.

REFERENCES


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