EXAMPLES OF PERIODIC MAPS ON EUCLIDEAN SPACES WITHOUT FIXED POINTS

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Let \( T \) be a map of period \( r \) on a Euclidean space \( E^n \). Smith seems to have been the first to consider fixed points of \( T \). He showed that \( T \) has a fixed point if \( r \) is a prime in [4], extended this result to \( r \) a power of a prime, and raised the question concerning the existence of a fixed point for \( r \) not a prime power in [5]; also cf. Problem 33 in [3]. Conner and Floyd gave an example of a contractible manifold \( M_r \) for every \( r \) not a prime power, and a map \( T \) of period \( r \) on \( M_r \) without fixed points [2]. They conjectured that \( M_r \) was a Euclidean space. This note shows that a slight modification of their example is Euclidean, hence:

**Theorem.** If \( r \) is an integer which is not a power of a prime, then there exists a triangulation \( \tau \) of \( E^{kr} \), a map \( T \) of period \( r \) on \( E^{kr} \) without fixed points, and \( T \) is simplicial relative to \( r \).

I wish to express my indebtedness to Professor Floyd for his help and encouragement.

**Preliminaries.** Let \( K \) be a subcomplex of a Euclidean space \( E \) under a triangulation \( \tau \). Let \( \sigma_K^{(0)} \) be the subdivision of \( \tau \) obtained by adding barycenters of all simples not contained in \( K \), cf. [6, p. 251]. \( \sigma_K^{(t+1)} = (\sigma_K^{(t)})^{(0)} \). If \( K \) is the empty complex, \( \sigma_K^{(0)} = \tau \), the usual \( i \)th barycentric subdivision. Denote the closed star of \( K \) in \( \tau \) by \( V(K, \tau) \) and let \( V^2(K, \tau) = V(V(K, \tau), \tau) \). \( N_W(K, \tau) = N(K, \tau) \) is a "regular" neighborhood of \( K \); cf. [6, p. 293]. If \( K \) is a contractible finite subcomplex having dimension \( m \) and \( E = E^n \), where \( n \geq 2m + 5 \), then it follows from Corollary 3 in [6, p. 298] that \( N_W(K, \tau) \) is an \( n \)-cell. Much use is made of this fact; however it will be convenient later to use the following neighborhood: \( N_1(K, \tau) = V(K^{(2)}, \tau^{(2)}) \), i.e. the star of \( K \) (subdivided twice barycentrically) in \( \tau^{(2)} \). Since it will be necessary to use Whitehead's result, but only in a topological way (i.e. noncombinatorial), it suffices to show that \( N_W(K, \tau) \) and \( N_1(K, \tau) \) are homeomorphic. This can be done by looking at an \( n \)-simplex \( p \) in the triangulation \( \sigma_K^{(0)} \) which intersects \( K \), and constructing a canonical homeomorphism of \( N_W(K, \tau) \cap p \) and \( N_1(K, \tau) \cap p \) in such a way that two such homeomorphisms match on \( p \)-faces, \( p < n \). Let \( \rho = \rho_0 \circ \rho_1 \).

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the join, where \( \rho_0 \) and \( \rho_1 \) are simplexes in \( \sigma^0 \) with \( \rho_0 \cap K = \emptyset \) and \( \rho_1 \subseteq K \). It is easy to show that every segment \( A_{x_0, x_1} = \{ x_t = (1-t)x_0 + tx_1 \mid t \in [0, 1] \} \) where \( x_0 \) is in \( \rho_0 \), \( x_1 \) in \( \rho_1 \), intersects \( N_W(K, \sigma) \) and \( N_i(K, \sigma) \) in a desirable way; namely there exist \( t_w \) and \( t_i \) in \( (0, 1) \) such that \( A_{x_0, x_1} \cap N_W(K, \sigma) = \{ x_t \mid t_w \leq t \leq 1 \} = A_w \) and \( A_{x_0, x_1} \cap N_i(K, \sigma) = \{ x_t \mid t_i \leq t \leq 1 \} = A_i \). Now map \( A_w \) onto \( A_i \) linearly and do this for each pair \( x_0, x_1 \) in \( \rho_0 \) and \( \rho_1 \). The regular neighborhoods used here will now be \( N_i \), and we define \( N_i(K, \sigma) = N_i(N_i(K, \sigma), \sigma^{(2)}) \), a subcomplex in \( E \) under \( \sigma^{(2i+2)} \). Clearly if \( K_i \) is a subcomplex under \( \sigma_i, K_1 \subseteq K, \) and \( \sigma_i \) refines \( \sigma \), then \( N_i(K_1, \sigma_i) \subseteq N_i(K, \sigma) \), for all \( i \).

For a given \( r \) in the theorem let \( K \) be the Conner-Floyd example of a 4-dimensional, star-finite, contractible complex with \( Z_r \) acting without fixed points [2, p. 360]. \( K \) is the union of simplicial mapping cylinders \( C_i \) whose “beginning” \( B_i \) and “end” \( E_i \) are 3-spheres. \( C_i \) is the simplicial analogy of the ordinary mapping cylinder defined, in this case, by a simplicial map \( f^* \) of \( B_i \) into \( E_i \) which is inessential. Also \( E_i = B_{i+1} \). For a more exact description the reader is referred to [2]. It will also be assumed that \( K \) is imbedded as a subcomplex of \( E = E^r \) with triangulation \( \sigma \), \( S \) is a simplicial map of \( E \) onto itself of period \( r \) (in fact, \( S(x_1, \ldots, x_r) = (x_2, \ldots, x_r, x_1) \) for \( x_i \subseteq E^r \)), \( K \) is an invariant set under \( S \), and \( S \mid K \) is the generator of \( Z_r \). This, too, was done in [2]. \( \sigma \) can be taken fine enough so that \( V(C_i, \sigma) \) and \( V(C_j, \sigma) \) are disjoint if \( |i-j| > 1 \). Since \( S \) is simplicial and \( K \), invariant, \( N_i(K, \sigma) \) is invariant for each \( i \), hence \( E' = \bigcup_{i=1}^{\infty} N_i(K, \sigma) \) is invariant. Since \( S \) has no fixed points in \( K \) and \( E' \subseteq \text{Int} \ V(K, \sigma) \), \( S \) has no fixed points in \( E' \). It will be shown that \( E' \) is Euclidean. To do this it will suffice to express \( E' \) as the union of cubes \( \{ I_j \}_{j=1}^{\infty} \) with \( I_j \subseteq \text{Int} I_{j+1} \) and then one could use a recently announced result of M. Brown. However, in this special case, where each \( I_j \) is in \( E^n \), \( \bigcup_{j=1}^{\infty} I_j \) is seen to be Euclidean by an easy application of another result of Brown which is in the literature [1]. For by the characterization of a tame \( S^{n-1} \) in \( E^n \) given there, and by taking \( I_j \) to be a slightly smaller concentric cube, there is no loss of generality in assuming that \( \text{Bd} \ I_j \) is a tame \( S^{n-1} \) in \( E^n \) for each \( j \). Hence the complement \( J \) of \( \bigcup_{j=1}^{\infty} I_j \) in the one point compactification of \( E^n \) is the intersection of decreasing cubes, i.e. cellular [1], hence \( E^n = S^n - J = (E^n \cup \infty) - J = \bigcup_{j=1}^{\infty} I_j \).

**Lemma.** Given any positive integer \( i \) there exists a subdivision \( \sigma_i \) of \( \sigma \) and a finite contractible complex \( K_i \) in \( \sigma_i \) such that

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2 The monotone union of open \( n \)-cells is an open \( n \)-cell, Notices Amer. Math. Soc. vol. 7 (1960) p. 478.

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(1) \( L_i = \bigcup_{j \in i} C_j \subseteq K_i \).
(2) \( \sigma_j \) agrees with \( \sigma \) on \( V(L_i, \sigma) \), hence \( N_j(L_i, \sigma) \subseteq N_j(K_i, \sigma_i) \), for all \( j \).
(3) \( N_j(K_i, \sigma_i) \subseteq N_j(L_{i+2}, \sigma) \), for all \( j \).

PROOF. Note that if \( D \) is a 4-cell and \( h \) is a homeomorphism from the \( \text{Bd} \ D \) onto the end \( E_{i+1} \) of \( C_{i+1} \), then since \( E_{i+1} \) is a strong deformation retract of \( C_{i+1} \), hence of \( L_{i+1} \), it follows that the identification space \( L_{i+1} \cup D / h \) is contractible. The proof of the lemma depends on getting a simplicial representation \( K_i \) of this identification space close to \( L_{i+2} \).

First we produce a map \( f \) of \( D \) into \( C_{i+2} \). For a simplicial model of \( D \) take the cone over the simplicial 3-sphere \( B \), where \( B \) is a copy of \( B_{i+2} \), the beginning of \( C_{i+2} \). Let \( D' = \{(b, t) \in D \mid t \in [0, 1/2]\}, D'' = \{(b, t) \in D \mid t \in [1/2, 1]\}. \) \( D' \cap D'' = B' \). Now map \( D' \) into \( C_{i+2} \) by \( f' \) such that \( f'| B : B \rightarrow B_{i+2} \) is a simplicial isomorphism, \( f'(B') \subseteq E_{i+2} \) and \( f'| B' \) is inessential (it can be taken to be, essentially, \( f^* \)). Hence there is a map \( f'' : D'' \rightarrow E_{i+2} \) such that \( f''| B' = f'| B' \). Then \( f(D') = f' \) and \( f(D'') = f'' \) define \( f \).

Let \( \varepsilon \) be so small that the \( \varepsilon \)-neighborhood (under the usual metric for \( E \)) of \( C_{i+2} \) is contained in \( N_1(C_{i+2}, \sigma) \). Since \( \dim E = 9r > 8 \) we can get \( g \), an \( \varepsilon \)-approximation of \( f \), which imbeds \( D \) in \( E \) (hence in \( N_1(C_{i+2}, \sigma) \)) and such that

(a) \( g| B = f| B \) and \( g(D) \cap L_{i+1} = B_{i+2} = g(B) \),
(b) \( g \) maps linearly (using the vector space structure of \( E \)) each simplex in the \( k \)th barycentric subdivision of \( D \), for some \( k \).

The usual technique is used, that of subdividing \( D \) so that images of simplexes under \( f \) are small relative to \( \varepsilon \), then choosing a point near each image of a vertex (keeping fixed images of vertices in \( B \)) so that the set of all such points is in general position, and then extending the obvious vertex map linearly.

Now we get a subdivision of \( V(C_{i+2}, \sigma) \) so that \( g(D) \) may be regarded as a subcomplex. One way of getting this would be to regard each 4-simplex in \( g(D) \) as a subsimplex of a rectilinear \( n \)-simplex in \( E \). Consider the \( (n-1) \)-planes determined by the \( (n-1) \)-faces of such \( n \)-simplexes, one chosen for each 4-simplex in \( g(D) \). The triangulation \( \sigma_i \), together with this finite collection of \( (n-1) \)-planes, partitions \( V(C_{i+2}, \sigma) \) into convex polyhedral sets which can then be triangulated. Furthermore this triangulation \( \sigma_i \) can be taken so fine that \( N_1(g(D), \sigma_i) \subseteq N_1(C_{i+2}, \sigma) \). Now extend the triangulation \( \sigma_i \) to all of \( E \) keeping \( \sigma \) on \( F = \text{Cl}(E - V^2(C_{i+2}, \sigma)) \). This can be done by triangulating the "ring" \( R = \text{Cl}(V^2(C_{i+2}, \sigma) - V(C_{i+2}, \sigma)) \) without introducing any new vertices in \( V^2(C_{i+2}, \sigma) - V(C_{i+2}, \sigma) \). Each simplex under \( \sigma \) in \( R \), say \( \rho \), may be regarded as a join of two simplexes \( \rho_1 \) and \( \rho_2 \) in \( F \) and
$V(C_{i+2}, \sigma)$ respectively, where $\rho_2$ has been subdivided under $\sigma_i$. Triangulate $\rho_1$ by taking the joins of $\rho_1$ with the small simplexes in $\rho_2$ under $\sigma_i$. Do this for each $\rho$ in $R$ getting a triangulation $\sigma_i$ of $E$ which is a subdivision of $\sigma$, which agrees with $\sigma$ on $F$. Condition (2) of the lemma follows from $V(L_i, \sigma) \subseteq F$.

Define $K_i = L_{i+1} \cup g(D)$, a contractible subcomplex under the triangulation $\sigma_i$. Condition (1) is clearly satisfied. Since $\sigma_i$ refines $\sigma$, $N_i(L_{i+1}, \sigma_i) \subseteq N_i(L_{i+1}, \sigma)$ and since $N_i(g(D), \sigma_i) \subseteq N_i(C_{i+2}, \sigma)$, $N_i(g(D), \sigma_i) \subseteq N_i(C_{i+2}, \sigma)$ and it follows that $N_i(L_i, \sigma_i) \subseteq N_i(L_{i+1}, \sigma)$, $N_i(g(D), \sigma_i) \subseteq N_i(L_{i+1}, \sigma)$, hence condition (3) is satisfied and the lemma proved.

**Proof of the theorem.** Using the notation of the lemma, $N_i(K_i, \sigma_i)$ is an $n$-cell by [6], hence $N_i(K_i, \sigma_i)$, which can be expressed as $N_i(L_i+1(K_i, \sigma_i), \sigma_i(2i-2)) \approx N_i(L_i+1(K_i, \sigma_i), \sigma_i(2i-2))$, is an $n$-cell, which we designate by $I_i$. Using the lemma we get:

$$N_i(L_i, \sigma) \subseteq I_1 \subseteq N_1(L_2, \sigma) \subseteq N_2(L_3, \sigma) \subseteq I_3 \subseteq N_i(L_5, \sigma) \subseteq N_5(L_6, \sigma) \subseteq I_5 \subseteq \cdots$$

and $I_{2i-1}$ is contained in the interior of $I_{2i+1}$. Then $E' = \bigcup_{i=1}^{\infty} N_i(K, \sigma) = \bigcup_{i=1}^{\infty} N_i(L_i, \sigma) = \bigcup_{i=1}^{\infty} I_{2i-1}$ is Euclidean. The map $T$ is, of course, $S|E'$, and the invariant triangulation is obtained in the following way. $N_i(K, \sigma)$ is a complex in $\sigma(2i)$, and it is subdivided twice without subdividing $N_{i-1}(K, \sigma)$, i.e. $N_i(K, \sigma)$ becomes a complex in $\sigma(2i)^2N_{i-1}(K, \sigma)(2i)$. 

*Added in proof.* D. R. McMillan has communicated to me an alternate (and simpler) way of producing examples in $E^{18r}$ making use of recent results of his in *Cartesian products of contractible open manifolds*, Bull. Amer. Math. Soc., this issue, pp. 510-514

**References**