CONICAL SINGULAR POINTS OF DIFFEOMORPHISMS

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1. Introduction. The Schoenflies extension $\Lambda_\phi$ of a differentiable mapping $\phi$, constructed in the proof of Theorem 2.1 of [1], has at most a differential singularity of conical type (to be defined). This fact has far-reaching consequences which are reflected in the theorems of [2]. Theorem 1.1 below is one of these consequences. No proof of Theorem 1.1 is given here.

Let $S$ be an $(n-1)$-sphere in a euclidean $n$-space $E$ and let $JS$ be the closed $n$-ball in $E$ bounded by $S$.

**THEOREM 1.1.** Let $z$ be an arbitrary point of $S$. A real analytic diffeomorphism $f$ of $S$ into $E$ admits a homeomorphic extension, $F$, defined over a set $Z \cup \{z\}$, where $Z$ is some open neighborhood of $JS - z$, and $F|Z$ is a real analytic diffeomorphism of $Z$ into $E$.

This extension $F$ of $f$ defines an analytic diffeomorphism of its domain of definition with $z$ deleted, and a homeomorphism with $z$ included. $F$ has no singularity on the interior of $S$, or on $S$, except at most at $z$.

We continue with a detailed exposition leading to a proof of Theorem 2.1.

**NOTATION.** Let $E$ be the euclidean $n$-space of points (or vectors) $x$ with rectangular coordinates $(x_1, \ldots, x_n)$. Let $\|x\|$ be the distance of $x$ from the origin $O$. Set

\[(1.1) \quad S = \{x \mid \|x\| = 1\}.\]

If $M$ is a topological $(n-1)$-sphere in $E$, $\bar{M}$ shall denote the open interior of $M$. The complement of a subset $Y$ of $E$ will be denoted by $CY$. We use $\text{diff}$ as an abbreviation of diffeomorphism.

A $C^n_\text{diff}$, $m > 0$. Let $x \mapsto G(x)$ be a homeomorphism into $E$ of an open neighborhood $X$ of a point $z \in E$; if $G| (X - z)$ is a $C^m$-diff into $E$, $G$ will be called a $C^n_\text{diff}$ of $X$ into $E$.

An admissible cone $K_z$. Let $K_z$ be a closed $n$-cone in $E$ with vertex $z$, and with sections orthogonal to $A$ which are closed $(n-1)$-balls whose centers are on $A$. The cone $K_z$ is determined by $z$, $A$ and any one of its orthogonal sections meeting $A - z$.

A conical point $z$ of $G$. Let $G$ be a $C^n_\text{diff}$ into $E$ of an open neighborhood $X$ of $z$. The point $z$ will be said to be a conical point of $G$ and
**K, a cone of singular approach** to z if there exists a \( C^m \)-diff \( \xi \) into \( E \) of some open neighborhood \( U \subset X \) of \( z \) such that
\[
G(x) = \xi(x) \quad (x \in U - K_z).
\]

On the supposition that \( z \) is a conical point of \( G \) we prove the following lemma.

**Lemma 1.1.** (i) If \( \mu \) is a \( C^m \)-diff into \( X \) of an open neighborhood \( Y \) of a point \( y \) such that \( \mu(y) = z \), then \( G\mu \) is a \( C^m \)-diff of \( Y \) into \( E \) with conical point \( y \).

(ii) If \( \theta \) is a \( C^m \)-diff of \( G(X) \) into \( E \), then \( \theta G \) is a \( C^m \)-diff of \( X \) into \( E \) with conical point \( z \).

**Proof of (i).** Suppose that \( G \) is represented on \( U - K_z \) as in (1.2). Let \( W \subset Y \) be an open neighborhood of \( y \) so small that \( \mu(W) \subset U \), and for some admissible cone \( K_y \)
\[
\mu(W - K_y) \subset U - K_z.
\]
Put \( \mu|_W = \mu_1 \). Then \( \xi \mu_1 \) is a \( C^m \)-diff of \( W \) into \( E \), and it follows from (1.2) and (1.3) that
\[
(G\mu)(x) = (\xi \mu_1)(x) \quad (x \in W - K_y).
\]

This partial representation (1.4) of \( G\mu \) shows that \( y \) is a conical point of the \( C^m \)-diff \( G\mu \). This establishes (i).

The proof of (ii) is immediate.

**2. The principal theorem.** In [1] there is given a \( C^m \)-diff \( \phi \) into \( E \) of a “shell” neighborhood \( \delta \) of \( S \) such that \( \phi \) carries points of \( \delta \) interior to \( S \) into points of \( E \) interior to the manifold \( \phi(S) \), and it is shown (see [1, Theorem 2.1]) that there exists an open neighborhood \( U \subset \delta \) of \( S \), a point \( z \in \mathcal{J}S \) and a \( C^m \)-diff \( \Lambda_\phi \) of \( U \cup \mathcal{J}S \) into \( E \) which extends \( \phi|_U \). The construction of \( \Lambda_\phi \) is carried through in [1] for the case in which \( \phi \) is special, in the sense that \( \phi \) reduces to the identity in the neighborhood of a point \( Q \) of \( S \). In this paper we supplement Theorem 2.1 of [1] by proving the following.

**Theorem 2.1.** The \( C^m \)-diff \( \Lambda_\phi \), as constructed in [1] for a “special” \( C^m \)-diff \( \phi \), has \( z \) as conical point.

To prove this theorem we review the necessary parts of the construction of \( \Lambda_\phi \) in [1].

**The relevant subsets of \( E \).** Let \( K \) be the open \( n \)-cube [1, p. 273]
\[
K = \{ x \mid -1 < x_i < 1; \ i = 1, \ldots, n \}.
\]
Let $K'$ be the subrectangle of $K$ on which $x_n < 0$. Subrectangles $H' \supset L' \supset G'$ of $K'$ are introduced with faces parallel to those of $K'$, of which $H'$ and $L'$ are open and $G'$ closed, while

$$\text{Cl } H' \subset K', \quad \text{Cl } L' \subset H'. \tag{2.2}$$

Let $D = \{ x \mid -1 < x_i < 9; \ i = 1, \cdots, n \}$ and set $P = (8, 0, \cdots, 0)$.

A radial mapping $R$ of $E$ onto $E$. $R$ is defined by the equations

$$y_1 - 8 = \frac{x_1 - 8}{2}; \quad y_j = \frac{x_j}{2} \quad (j = 2, \cdots, n) \tag{2.3}$$

and leaves $P$ fixed. If $R^r$ is the $r$-fold iterate of $R$ and $R^0$ the identity, the space $E$ admits a trivial partition

$$E = \bigcup_{r=0}^{\infty} R^r(K) \cup P \cup A \quad \text{(Cf. (5.1) of [1])} \tag{2.4}$$

provided $A$ is suitably chosen.

The mapping $T$. If $B$ is a bounded subset of $E$, $\text{Int } B$ shall denote the smallest $n$-rectangle $\Pi$ in $E$ with faces parallel to the coordinate $(n-1)$-planes and such that $\Pi \supset B$. In §6 of [1], a $C^\infty$-diff $T$ of $E$ onto $E$ is defined. For us the essential properties of $T$ are that

$$RT(K) \cap T(K) = \varnothing, \quad T(K) \subset \text{Int}(K \cup RK). \tag{2.5}$$

One sets $T_{r+1} = R^r T$, $r = 0, 1, \cdots$.

The cone $K_P$. Let $K_P$ be the smallest admissible cone with vertex $P$, with axis the segment of the $x_1$-axis on which $x_1 \leq 8$, and with $K_P \supset \text{Int}(K \cup RK)$. One sees that

$$K_P \supset T_r(K) \cup G' \quad (r = 1, 2, \cdots). \tag{2.6}$$

The contraction $\alpha$. This is a $C^\infty$-diff of $D$ onto $H'$ which leaves $L'$ pointwise invariant [1, p. 274]. We infer that $\alpha(P) \in H' - \text{Cl } L'$.

The reflection $t$. The point $Q$ is the intersection of the positive $x_n$-axis with $S$. Let $S_Q$ be an $(n-1)$-sphere with center $Q$ and diameter $p < 1$, so small that $\phi|JS_Q$ reduces to the identity. Let $t$ be the reflection of $E - Q$ in $S_Q$ [1, p. 272].

The $C^m$-diff $\omega$. The domain of definition of $\omega$ includes $H' - G'$, and so is an open neighborhood of $\alpha(P)$. Cf. p. 273 of [1].

The mapping $\omega$. By Lemma 5.1 of [1], the domain of $\omega$ includes $A$, and $\omega(x) = x$ on $A$.

The mapping $\sigma$. By Lemma 7.2 of [1] $\sigma$ is a $C^m$-diff of $CG'$ into $E$. By this lemma and (2.6), $\sigma(x) = \omega(x)$ for $x \in CK_P$. Since $CK_P \subset A$ by (2.4), $\sigma(x) = x$ for $x \in CK_P$. Hence $P$ is a conical point of $\sigma$. 

The mapping $\lambda_a$. For present purposes $\lambda_a$ is a mapping for which (cf. (3.6) of [1])

$$
(\lambda_a)(x) = \omega(a(\sigma^{-1}(x))) \quad (x \in \sigma(D - G')).
$$

We verify that $P$ is a conical point of $\lambda_a$. The domain of $x$ in (2.7) is open. It contains $P$ since $P \in D - G'$ and $\sigma(P) = P$. Now $\sigma^{-1}$ maps $\sigma(D - G')$, as a $C^p$-diff, onto $D - G'$, with $P$ a conical point of $\sigma^{-1}$. Moreover $a(D - G') = H' - G'$, while $\omega$ operates as a $C^m$-diff on $H' - G'$. Returning to (2.7) observe that $\lambda_a$ defines a $C^p$-diff of $\sigma(D - G')$ into $E$. It follows from Lemma 1.1 (ii) and (2.7) that $P$ is a conical point of $\lambda_a$.

Completion of Proof of Theorem 2.1. In accord with the line following (7.19) of [1] and line -13 on p. 275 in [1], $z = t(a(P))$. By (7.22) of [1]

$$
(2.8) \quad \Lambda_\phi(x) = t(\lambda_a(t(x))) \quad (x \in t(H'))
$$

(cf. [1, lines 2–3, p. 287]) so that if one sets $\mu(x) = a^{-1}(t(x))$ for $x \in t(H')$

$$
(2.9) \quad \Lambda_\phi(x) = [t(\lambda_a)\mu](x) \quad (x \in t(H')).
$$

We now apply Lemma 1.1. The $C^\omega$-diff $x \mapsto \mu(x)$ of $t(H')$ into $E$ maps $z$ into $P$, since $z = t(a(P))$. From Lemma 1.1 and (2.9) we can then infer that $z$ is a conical point of $\Lambda_\phi$, since $P$ is a conical point of $\lambda_a$.

This establishes Theorem 2.1.

The generality of singularities of conical type is evidenced by the following theorem. Cf. [2].

Theorem 2.2. Let $F$ be an arbitrary $C^\omega$-diffeomorphism into $E$ of an open subset $X \subset E$. There exists a $C^\omega$-diffeomorphism $F^*$ of $X$ into $E$ for which $z$ is a conical point and which is such that $F^*(x) = F(x)$ except at most in an arbitrarily small prescribed neighborhood of $z$.

References


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