1. Introduction. The Schoenflies extension $A_\phi$ of a differentiable mapping $\phi$, constructed in the proof of Theorem 2.1 of [1], has at most a differential singularity of conical type (to be defined). This fact has far-reaching consequences which are reflected in the theorems of [2]. Theorem 1.1 below is one of these consequences. No proof of Theorem 1.1 is given here.

Let $S$ be an $(n - 1)$-sphere in a euclidean $n$-space $E$ and let $JS$ be the closed $n$-ball in $E$ bounded by $S$.

**Theorem 1.1.** Let $z$ be an arbitrary point of $S$. A real analytic diffeomorphism $f$ of $S$ into $E$ admits a homeomorphic extension, $F$, defined over a set $Z \cup z$, where $Z$ is some open neighborhood of $JS - z$, and $F|Z$ is a real analytic diffeomorphism of $Z$ into $E$.

This extension $F$ of $f$ defines an analytic diffeomorphism of its domain of definition with $z$ deleted, and a homeomorphism with $z$ included. $F$ has no singularity on the interior of $S$, or on $S$, except at most at $z$.

We continue with a detailed exposition leading to a proof of Theorem 2.1.

**Notation.** Let $E$ be the euclidean $n$-space of points (or vectors) $x$ with rectangular coordinates $(x_1, \cdots, x_n)$. Let $\|x\|$ be the distance of $x$ from the origin $O$. Set

\[(1.1) \quad S = \{x : \|x\| = 1\}.\]

If $M$ is a topological $(n - 1)$-sphere in $E$, $\mathcal{J}M$ shall denote the open interior of $M$. The complement of a subset $Y$ of $E$ will be denoted by $CY$. We use $\text{diff}$ as an abbreviation of diffeomorphism.

A $C^m$-diff, $m > 0$. Let $x \mapsto G(x)$ be a homeomorphism into $E$ of an open neighborhood $X$ of a point $z \in E$; if $G|(X - z)$ is a $C^m$-diff into $E$, $G$ will be called a $C^m$-diff of $X$ into $E$.

An admissible cone $K_z$. Let $K_z$ be a closed $n$-cone in $E$ with vertex $z$, and with sections orthogonal to $A$ which are closed $(n - 1)$-balls whose centers are on $A$. The cone $K_z$ is determined by $z$, $A$ and any one of its orthogonal sections meeting $A - z$.

A conical point $z$ of $G$. Let $G$ be a $C^m$-diff into $E$ of an open neighborhood $X$ of $z$. The point $z$ will be said to be a conical point of $G$ and
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$K_z$, a cone of singular approach to $z$ if there exists a $C^m$-diff $\xi$ into $E$ of some open neighborhood $U \subset X$ of $z$ such that

$$G(x) = \xi(x) \quad (x \in U - K_z).$$

On the supposition that $z$ is a conical point of $G$ we prove the following lemma.

**Lemma 1.1.** (i) If $\mu$ is a $C^m$-diff into $X$ of an open neighborhood $Y$ of a point $y$ such that $\mu(y) = z$, then $G\mu$ is a $C^m_y$-diff of $Y$ into $E$ with conical point $y$.

(ii) If $\theta$ is a $C^m$-diff of $G(X)$ into $E$, then $\theta G$ is a $C^m_y$-diff of $X$ into $E$ with conical point $z$.

**Proof of (i).** Suppose that $G$ is represented on $U - K_z$ as in (1.2). Let $W \subset Y$ be an open neighborhood of $y$ so small that $\mu(W) \subset U$, and for some admissible cone $K_y$

$$\mu(W - K_y) \subset U - K_z.$$  

Put $\mu|_W = \mu_1$. Then $\xi_\mu_1$ is a $C^m$-diff of $W$ into $E$, and it follows from (1.2) and (1.3) that

$$(G\mu)(x) = (\xi_\mu_1)(x) \quad (x \in W - K_y).$$

This partial representation (1.4) of $G\mu$ shows that $y$ is a conical point of the $C^m_y$-diff $G\mu$. This establishes (i).

The proof of (ii) is immediate.

2. **The principal theorem.** In [1] there is given a $C^m$-diff $\phi$ into $E$ of a "shell" neighborhood $\delta_\alpha$ of $S$ such that $\phi$ carries points of $\delta_\alpha$ interior to $S$ into points of $E$ interior to the manifold $\phi(S)$, and it is shown (see [1, Theorem 2.1]) that there exists an open neighborhood $U \subset \delta_\alpha$ of $S$, a point $z \in \partial S$ and a $C^m_\alpha$-diff $\Lambda_\phi$ of $U \cup \partial S$ into $E$ which extends $\phi|_U$. The construction of $\Lambda_\phi$ is carried through in [1] for the case in which $\phi$ is special, in the sense that $\phi$ reduces to the identity in the neighborhood of a point $Q$ of $S$. In this paper we supplement Theorem 2.1 of [1] by proving the following.

**Theorem 2.1.** The $C^m_\alpha$-diff $\Lambda_\phi$, as constructed in [1] for a "special" $C^m$-diff $\phi$, has $z$ as conical point.

To prove this theorem we review the necessary parts of the construction of $\Lambda_\phi$ in [1].

**The relevant subsets of $E$.** Let $K$ be the open $n$-cube [1, p. 273]

$$K = \{x^i \mid -1 < x^i < 1; i = 1, \cdots, n\}.$$
Let $K'$ be the subrectangle of $K$ on which $x_n < 0$. Subrectangles $H' \supset L' \supseteq G'$ of $K'$ are introduced with faces parallel to those of $K'$, of which $H'$ and $L'$ are open and $G'$ closed, while

\begin{equation}
\text{Cl } H' \subset K', \quad \text{Cl } L' \subset H'.
\end{equation}

Let $D = \{ x \mid -1 < x_i < 9; \ i = 1, \ldots, n \}$ and set $P = (8, 0, \ldots, 0)$. A radial mapping $R$ of $E$ onto $E$. $R$ is defined by the equations

\begin{equation}
y_1 - 8 = \frac{x_1 - 8}{2}; \quad y_j = \frac{x_j}{2} \quad (j = 2, \ldots, n)
\end{equation}

and leaves $P$ fixed. If $R^r$ is the $r$-fold iterate of $R$ and $R^0$ the identity, the space $E$ admits a trivial partition

\begin{equation}
E = \bigcup_{r=0}^{\infty} R^r(K) \cup P \cup A \quad (\text{Cf. (5.1) of [1]})
\end{equation}

provided $A$ is suitably chosen.

The mapping $T$. If $B$ is a bounded subset of $E$, $\text{Int } B$ shall denote the smallest $n$-rectangle $\Pi$ in $E$ with faces parallel to the coordinate $(n-1)$-planes and such that $\Pi \supset B$. In §6 of [1], a $C^\infty$-diff $T$ of $E$ onto $E$ is defined. For us the essential properties of $T$ are that

\begin{equation}
RT(K) \cap T(K) = \emptyset, \quad T(K) \subset \text{Int}(\overline{K} \cup R\overline{K}).
\end{equation}

One sets $T_{r+1} = R^r T$, $r = 0, 1, \ldots$.

The cone $K_P$. Let $K_P$ be the smallest admissible cone with vertex $P$, with axis the segment of the $x_1$-axis on which $x_1 \leq 8$, and with $K_P \supset \text{Int}(\overline{K} \cup R\overline{K})$. One sees that

\begin{equation}
K_P \supset T_r(K) \cup G' \quad (r = 1, 2, \ldots).
\end{equation}

The contraction $a$. This is a $C^\infty$-diff of $D$ onto $H'$ which leaves $L'$ pointwise invariant [1, p. 274]. We infer that $a(P) \in H' - \text{Cl } L'$.

The reflection $t$. The point $Q$ is the intersection of the positive $x_n$-axis with $S$. Let $S_Q$ be an $(n-1)$-sphere with center $Q$ and diameter $p < 1$, so small that $\phi|_{JS_Q}$ reduces to the identity. Let $t$ be the reflection of $E - Q$ in $S_Q$ [1, p. 272].

The $C^n$-diff $\omega$. The domain of definition of $\omega$ includes $H' - G'$, and so is an open neighborhood of $a(P)$. Cf. p. 273 of [1].

The mapping $\omega$. By Lemma 5.1 of [1], the domain of $\omega$ includes $A$, and $\omega(x) = x$ on $A$.

The mapping $\sigma$. By Lemma 7.2 of [1] $\sigma$ is a $C^\infty$-diff of $CG'$ into $E$. By this lemma and (2.6), $\sigma(x) = \omega(x)$ for $x \in CK_P$. Since $CK_P \subset A$ by (2.4), $\sigma(x) = x$ for $x \in CK_P$. Hence $P$ is a conical point of $\sigma$. 

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The mapping $\lambda \omega a$. For present purposes $\lambda \omega a$ is a mapping for which (cf. (3.6) of [1])

\begin{equation}
(\lambda \omega a)(x) = \omega(a(\sigma^{-1}(x))) \quad (x \in \sigma(D - G')).
\end{equation}

We verify that $P$ is a conical point of $\lambda \omega a$. The domain of $x$ in (2.7) is open. It contains $P$ since $P \subseteq D - G'$ and $\sigma(P) = P$. Now $\sigma^{-1}$ maps $\sigma(D - G')$, as a $C^p$-diff, onto $D - G'$, with $P$ a conical point of $\sigma^{-1}$. Moreover $a(D - G') = H' - G'$, while $\omega$ operates as a $C^m$-diff on $H' - G'$. Returning to (2.7) observe that $\lambda \omega a$ defines a $C^p$-diff of $\sigma(D - G')$ into $E$. It follows from Lemma 1.1 (ii) and (2.7) that $P$ is a conical point of $\lambda \omega a$.

**Completion of proof of Theorem 2.1.** In accord with the line following (7.19) of [1] and line -13 on p. 275 in [1], $s = t(a(P))$. By (7.22)" of [1]

\begin{equation}
\Delta_{\phi}(x) = t(\lambda \omega(t(x))) \quad (x \in t(H'))
\end{equation}

(cf. [1, lines 2–3, p. 287]) so that if one sets $\mu(x) = a^{-1}(t(x))$ for $x \in t(H')$

\begin{equation}
\Delta_{\phi}(x) = [t(\lambda \omega a)\mu](x) \quad (x \in t(H')).
\end{equation}

We now apply Lemma 1.1. The $C^m$-diff $x \mapsto \mu(x)$ of $t(H')$ into $E$ maps $z$ into $P$, since $z = t(a(P))$. From Lemma 1.1 and (2.9) we can then infer that $z$ is a conical point of $\Delta_{\phi}$, since $P$ is a conical point of $\lambda \omega a$.

This establishes Theorem 2.1.

The generality of singularities of conical type is evidenced by the following theorem. Cf. [2].

**Theorem 2.2.** Let $F$ be an arbitrary $C^p$-diffeomorphism into $E$ of an open subset $X \subseteq E$. There exists a $C^p$-diffeomorphism $F^*$ of $X$ into $E$ for which $z$ is a conical point and which is such that $F^*(x) = F(x)$ except at most in an arbitrarily small prescribed neighborhood of $z$.

**References**