COHOMOLOGY OF MAXIMAL IDEAL SPACES

BY ANDREW BROWDER

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Let $A$ be a commutative Banach algebra with unit, and let $M$ be the maximal ideal space of $A$. We say that $A$ is generated by $x_1, \ldots, x_n$ if the polynomials $p(x_1, \ldots, x_n)$ form a dense subalgebra of $A$. Let $H^j(M, C)$ denote the $j$th Čech cohomology group of $M$ with complex coefficients.

**Theorem.** If $A$ is generated by $n$ elements, then $H^j(M, C) = 0$ for $j \geq n$.

**Proof.** If $x_1, \ldots, x_n$ generate $A$, then the map of $M$ into $C^n$ given by $h \mapsto (h(x_1), \ldots, h(x_n))$ is a homeomorphism of $M$ onto a compact set $K$. It is known (see, e.g., [1]) that $K$ is polynomially convex, i.e., if $V$ is any open set containing $K$, there exists an analytic polyhedron $U$ defined by polynomials, such that $K \subset U \subset V$. Each such polyhedron $U$ is a domain of holomorphy (Stein manifold) and a Runge domain. For any $n$-dimensional Stein manifold $U$, it is known that $H^j(U, C) = 0$ for $j > n$. (See [2] for a proof.) For any Runge domain $U$ in $C^n$, Serre has shown [3] that $H^n(U, C) = 0$. The proof is completed by observing the following nonstandard but elementary continuity property of Čech cohomology:

**Fact.** Let $X$ be a compact subset of a metric space, $G$ an abelian group, $j$ a non-negative integer. If for every open set $V \supset K$, there exists an open $U$ with $K \subset U \subset V$ and $H^j(U, G) = 0$, then $H^j(K, G) = 0$.

**Corollary.** Let $M$ be an $n$-dimensional compact orientable manifold. Let $C(M)$ denote the ring of all continuous complex-valued functions on $M$, normed by the sup norm. Then $C(M)$ requires at least $n+1$ generators.

**Remarks.** 1. For $n=1$, the condition of the theorem is both necessary and sufficient; a compact subset $K$ of the plane is polynomially convex if and only if $K$ has connected complement, which is equivalent to $H^1(K, C) = 0$.

2. It is of course trivial that at least $n+1$ real-valued functions are required to generate $C(M)$ when $M$ is a compact $n$-dimensional manifold, but it should be observed that in general, a compact space $X$ need not require as many complex functions to generate $C(X)$ as it does real functions. Example: If $X$ is a compact connected plane set

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with no interior and connected complement, $C(X)$ is generated by the single function $z$ (Mergelyan's theorem); but $C(X)$ is generated by a single real function if and only if $X$ is a Jordan arc.

3. The author is unaware of any other proof of the corollary even for the case $M$ the two-sphere.

References


Brown University