STRONG RATIO LIMIT PROPERTY

BY STEVEN OREY[1]

Communicated by M. Loève, August 8, 1961

1. **Introduction.** For every nonnegative integer n let $p_{ij}^{(n)}$ be the n-step transition probabilities of a recurrent, irreducible, aperiodic Markov chain, $i, j=0, 1, \cdots$. We say the chain has the *strong ratio limit property* (*SRLP*) if there exist positive constants $\pi_j, j=0, 1, \cdots$, such that

$$(1)\qquad \lim_{n\to\infty} \frac{p_{ij}^{(n+m)}}{p_{kh}^{(n)}} = \frac{\pi_j}{\pi_h}, \qquad m = 0, \pm 1, \pm 2, \cdots.$$

It is well known that SRLP does not hold for all chains of the type considered here.[2] We here present conditions for SRLP; the continuous parameter case is also considered.

2. **Discrete parameter.** Let ${}_kp_{ij}^{(n)}=$ Prob [going from i to j in n steps without visiting k at step number $1, 2, \cdots, n-1$]. Note

$$(2.1)\qquad \sum_{n=1}^{\infty} {}_jp_{ij}^{(n)} = 1 \quad \text{and} \quad \text{g.c.d.}\{n: p_{ii}^{(n)} > 0\} = 1 \quad \text{for every } i, j.$$

LEMMA 1. SRLP *holds if and only if* $p_{00}^{(n+1)}/p_{00}^{(n)} \to 1$ *as* $n\to\infty$.

SKETCH OF PROOF. Assume $p_{00}^{(n+1)}/p_{00}^{(n)} \to 1$ as $n\to\infty$. For $n>N\geqq 1$ we have

$$(2.2)\qquad \begin{aligned} A_n(\alpha) &= p_{0\alpha}^{(n)}/p_{00}^{(n)} = \sum_{v=1}^{n} {}_0p_{0\alpha}^{(v)} p_{00}^{(n-v)}/p_{00}^{(n)} = \sum_{v=1}^{N} + \sum_{v=N+1}^{n} \\ &= B_{N,n}(\alpha) + C_{N,n}(\alpha). \end{aligned}$$

Observe $A_n(\alpha)$ converges as $n\to\infty$ if and only if

$$(2.3)\qquad \lim_{N\to\infty} \lim_{n\to\infty} C_{N,n}(\alpha)$$

---

[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research and Development Command, under Contract No. AF-49(638)-617. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[2] For counterexample, general discussion and references to the literature see [1] under "ratio limit theorem, individual."

Frequently authors consider only the case $m=0$ in relation to (1). We do not know whether this is really more restrictive or not.

exists. Setting $\alpha=0$ in (2.2) we see that the expression in (2.3) equals zero when $\alpha=0$. Let j be an integer, depending on α, such that ${}_0p_{\alpha 0}^{(j)}>0$. For $\alpha\neq 0$ the estimate ${}_0p_{00}^{(n+j)}\geqq {}_0p_{0\alpha}^{(n)}\,{}_0p_{\alpha 0}^{(j)}$ shows that (2.3) equals zero for every α, proving the convergence of $A_n(\alpha)$. Similar arguments establish the lemma in full generality.

Let $u_0=1$, $u_n=p_{00}^{(n)}$, $f_n={}_0p_{00}^{(n)}$, $n=1, 2, \cdots$. We have the familiar renewal relation

$$u_n = \sum_{k=1}^{n} f_k u_{n-k}, \qquad n = 1, 2, \cdots. \tag{2.4}$$

According to Lemma 1 SRLP is equivalent to

$$u_{n+1}/u_n \to 1 \text{ as } n \to \infty. \tag{2.5}$$

We shall need the following simple lemma proved in [3]:

LEMMA 2. *If* lim sup $(u_{n+1}/u_n)\leqq 1$ *as* $n\to\infty$ *then* (2.5) (*and therefore SRLP*) *holds.*

PROOF. Let $m=\liminf (u_{n+1}/u_n)$, $M=\limsup u_{n+1}/u_n$ as $n\to\infty$. From (2.4) one gets easily

$$m > \sum_{k=1}^{\infty} f_k M^{-k+1}. \tag{2.6}$$

Thus if $M\leqq 1$, $m\geqq M$ so that $M=m=1$.

THEOREM 1. *If for some positive integer* m

$$\limsup u_{m(n+1)}/u_{mn} \leqq 1 \quad \textit{as } n \to \infty \tag{2.7}$$

then (2.5) *holds.*

SKETCH OF PROOF. Assume (2.7). It is probabilistically evident that the sequence $\{u_{mn}\}$, $n=0, 1, \cdots$ is also a "renewal sequence" for some persistent, aperiodic, recurrent event, so that Lemma 2 applies to give $u_{m(n+1)}/u_{mn}\to 1$ as $n\to\infty$. Further probabilistic arguments lead to $u_{mn+j}/u_{mn}\to 1$ as $n\to\infty$ for $j=1, 2, \cdots, m-1$.[3]

The next theorem has several interesting probabilistic interpretations. However, the only proof we know is that given in our joint paper [3] which uses a fairly intricate analytic argument.

THEOREM 2. *Condition* (2.5) *is equivalent to*

$$\lim_{N\to\infty}\left[\sup_{n\geqq N}\frac{1}{u_n}\sum_{k=N}^{n} f_k u_{n-k}\right] = 0. \tag{2.8}$$

[3] A purely analytic proof of this theorem seems much harder. In [3] we give an analytic proof of a weaker theorem.

3. Continuous parameter. Let $p_{ij}(t)$ be the transition probability matrix of a continuous parameter Markov chain, $t \geqq 0, i, j = 0, 1, \cdots$. We assume $p_{ij}(t) \to \delta_{ij}$ as $t \to 0$. We take the chain to be irreducible and recurrent. The SRLP now becomes

$$\lim_{t\to\infty} \frac{p_{ij}(t+\Delta)}{p_{kh}(t)} = \frac{\pi_j}{\pi_h}, \qquad -\infty < \Delta < \infty. \tag{3.1}$$

We have

LEMMA 3. *If for some* $\Delta > 0$

$$\lim_{t\to\infty} \frac{p_{00}(t+\delta)}{p_{00}(t)} = 1 \text{ uniformly for } 0 \leqq \delta \leqq \Delta \tag{3.2}$$

then SRLP holds.

4. Reversible processes. The matrices $p_{ij}^{(n)}(p_{ij}(t))$ of §1 (§2) belong to a *reversible* process if and only if $\pi_i p_{ij} = \pi_j p_{ji}(\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ for all t) for every i and j.

THEOREM 3. *Reversibility implies SRLP, both in the discrete and continuous parameter case.*

PROOF. Assume reversibility. According to Kendall [**6**; **7**] we have the representation

$$p_{ij}^{(n)} = \int_{-1}^{1} x^n d\Psi_{ij}(x), \qquad n = 0, 1, \cdots, \tag{4.1}$$

in the discrete parameter case and

$$p_{ij}(t) = \int_{0}^{\infty} e^{-tx} d\Psi_{ij}(x), \qquad t \geqq 0, \tag{4.2}$$

in the continuous parmeter case, where the Ψ_{ij} are real-valued functions of bounded variation, nondecreasing when $i = j$.

Formula (4.1) shows $u_{2n}(=p_{00}^{(2n)})$ is nonincreasing so that SRLP follows from Theorem 1.

In the continuous parameter case the above argument gives us

$$p_{00}((n+1)\delta)/p_{00}(n\delta) \to 1 \text{ as } n \to \infty \tag{4.3}$$

for every $\delta > 0$. Formula (4.2) shows that $p_{00}(t)$ is nonincreasing. Thus condition (3.2) must hold, and Lemma 3 applies.

Representations like (4.1) and (4.2) have been used to establish SRLP in [**4**; **5**; **8**]. In previously treated cases however, the measures Ψ_{ij} were known more explicitly and the arguments depend on detailed investigation of the behavior of these measures. In [4; 5] there is also some discussion of SRLP in the transient case, which we have excluded. We do not know whether convergence of $p_{00}^{(n+1)}/p_{00}^{(n)}$ implies SRLP in the transient case.

BIBLIOGRAPHY

1. K. L. Chung, *Markov chains with stationary transition probabilities*, Berlin, Springer, 1960.

2. W. Feller, *An introduction to probability theory and its applications*, New York, Wiley, 1950.

3. A. Garsia, S. Orey, and E. Rodemich, *Asymptotic behavior of successive coefficients of some power series*, to appear.

4. S. Karlin and J. McGregor, *The classification of birth and death processes*, Trans. Amer. Math. Soc. vol. 86 (1957) pp. 366–400.

5. ———, *Random walks*, Illinois J. Math. vol. 3 (1959) pp. 66–81.

6. D. G. Kendall, *Unitary dilations of Markov transition operators, and the corresponding integral representation for transition-probability matrices*, Harold Cramér Volume (Ed. U. Grenander) Stockholm (1960) pp. 139–161.

7. ———, *Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states*, Proc. London Math. Soc. vol. 9 (1959) pp. 417–431.

8. W. Pruitt, *Bilateral birth and death processes*, ONR Technical Report No. 22, Contract Nonr-225(28) (NR-047-019), Applied Math. and Stat. Lab., Stanford, (1960).

UNIVERSITY OF MINNESOTA