1. Introduction. Fränkl and Pontrjagin [1] and Seifert [2] have shown that for any given family of disjoint polyhedral simple closed curves in three-space, there can always be found a polyhedral orientable surface in three-space whose boundary consists precisely of the given curves. The following theorem extends this result to surfaces in four-space.

Theorem 1. Let $M^2$ be a locally flat, polyhedral, closed orientable surface (not necessarily connected) in Euclidean four-space, $R^4$. Then there is an orientable polyhedral three-manifold, $M^3$, in $R^4$, whose boundary is $M^2$.

Local flatness means that for each vertex $v$ of $M^2$, the link of $v$ on $M^2$ (a simple closed curve) is unknotted in the link of $v$ in $R^4$ (a three-sphere). This condition is purely local and absolutely necessary. On the other hand, the restriction to orientable surfaces is required by the nature of the proof, and I do not know whether nonorientable surfaces of even characteristic in four-space bound nonorientable three-manifolds in four-space.

2. Outline of the proof. $M^3$ is first deformed so that its intersections with the horizontal hyperplanes $R^3_t = \{(x_1, x_2, x_3, x_4): x_4 = t\}$ are as simple as possible. What we have in mind is to find orientable surfaces in the $R^3_t$ whose boundaries are precisely $M^2 \cap R^3_t$, in such a continuous way that when considered together they form an orientable three-manifold $M^2$ whose boundary is $M^2$. The process is carried out with decreasing $t$, and the local flatness of $M^2$ assures us that the construction can be begun. As $t$ decreases, $M^2 \cap R^3_t$ changes isotopically, except at a finite number of singular values of $t$. A slight deformation of $M^3$ insures that we need only consider hyperbolic transformations, in which two arcs come together at a midpoint and then separate like the cross-sections of a saddle surface, and elliptic transformations, in which a simple closed curve shrinks to a point and then disappears (or vice versa). In the hyperbolic case, these arcs already form part

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2 Added in proof. This case is considered in a forthcoming paper.
of the boundary of a cross-sectional surface and the hyperbolic transformation could be extended to the surface in the natural way, except for the possibility of a number of sheets of the surface being in the way. These sheets are simply pierced one after the other with decreasing $t$, the cuts being joined as in [3, page 4], to preserve orientability of the cross-sections. Finally, when no more sheets are in the way, the original hyperbolic transformation is extended to the cross-sectional surface. That this final transformation does not destroy the orientability of the cross-sections requires a special argument.

In the case of the elliptic transformations, if with decreasing $t$ a point opens up into a simple closed curve (which must be unknotted by the local flatness of $M^3$), then we simply introduce another component of the cross-sectional surface which with decreasing $t$ opens up from a point into a two-cell. The serious case occurs when a component of $M^3 \cap R^3$ shrinks to a point and then disappears with decreasing $t$. Call this component $c_1$, and let $c_2, \ldots, c_k$ be the other boundary curves of the component $G$ of the cross-sectional surface containing $c_1$. Since $c_1$ is unknotted by the local flatness of $M^3$, let $D$ be a polyhedral two-cell in $R^3$ bounded by $c_1$. Let $c'$ be a simple closed curve on $G$ lying in a small neighborhood of $c_1$ and "parallel" to $c_1$. Because the cross-sectional surface, and hence $G$ is orientable, the linking number of $c'$ with $c_1$ is the same as the sum of the linking numbers of the $c_2, \ldots, c_k$ with $c_1$. But each of these linking numbers is zero, since $c_1$ is about to shrink to a point away from all these curves. Because the linking number of $c'$ with $c_1$ is zero, the cross-sectional surface can be deformed so that a small neighborhood of $c_1$ on $G$ meets $D$ only at $c_1$, while the total intersection of the cross-sectional surface with $D$ consists of a number of simple closed curves. Each of these intersections can be removed by standard hyperbolic transformations with decreasing $t$, until finally $D$ meets the cross-sectional surface only at its boundary curve $c_1$. By a slight deformation of $M^3$, the original elliptic transformation can be altered so as to shrink $c_1$ to a point along $D$, closing up a component of the cross-sectional surface and completing the construction for the elliptic transformation. When finally $t$ has decreased below the minimum value attained by the fourth coordinates of points of $M^3$, the cross-sectional surface consists of a number of closed orientable surfaces in a three-dimensional hyperplane $R^3$. It remains to shrink off the components of this surface to points with decreasing $t$. If the resulting $M^3$ is to be a manifold, this must be done by first changing these components into two-spheres. But R. H. Fox has shown in [4, Theorem 2] that whenever we are given a number of polyhedral...
closed orientable surfaces in three-space, not all of which are two-
spheres, a hyperbolic transformation may be found which either de­
creases the total genus or else increases the number of components
with positive genus while leaving the total genus unaltered. We carry
out such a transformation with decreasing $t$, and repeat the procedure
until all the components of the cross-sectional surface are two-
spheres, which may then be shrunk to points as $t$ decreases further,
completing the construction of $M^3$.

As the various transformations undergone by the cross-sectional
surfaces are topologically equivalent to those experienced by a cross-
section of a hypersurface in $R^4$ (with due regard being taken of the
fact that $M^3$ has a boundary), it is easily seen that $M^3$ is a manifold.
Furthermore, since the cross-sections are orientable and the various
transformations preserve orientations, $M^3$ is also orientable.

R. H. Bing has pointed out to me that if we are willing to allow a
three-dimensional cross-section, then the argument can be completed
as soon as the cross-sectional surface becomes closed, for every closed
surface in three-space, whether connected or not, is the boundary of
a three-dimensional region.

References

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