The purpose of this note is to announce results obtained in the analytic continuation of the (nondegenerate) "principal series" of representations of the $n \times n$ complex unimodular group. This study has as its starting point a similar one for the $2 \times 2$ real unimodular group previously carried out by us in [4].

We let $G$ be the $n \times n$ complex unimodular group and $C$ its diagonal subgroup consisting of elements $c = (c_1, c_2, \cdots, c_n)$. A continuous character $\lambda$ of $C$ is given by

$$\lambda(c) = \left( \frac{c_1}{c_1} \right)^{m_1} \cdots \left( \frac{c_n}{c_n} \right)^{m_n} |c_1|^{s_1} \cdots |c_n|^{s_n}$$

where the sequences of integers $m_1, m_2, \cdots, m_n$ and complex numbers $s_1, s_2, \cdots, s_n$ are uniquely determined by setting $0 \leq m_1 + m_2 + \cdots + m_n < n$ and $s_1 + s_2 + \cdots + s_n = 0$. Notice that $\lambda$ is unitary, i.e., has values in the circle group, if $\text{Re}(s_j) = 0$, $j = 1, 2, \cdots, n$. Gelfand and Neumark have shown how to construct for each unitary $\lambda$ an irreducible unitary representation $a \mapsto T(a, \lambda)$ of the group $G$ [2]. To describe these representations (i.e., the principal series) we follow the method but not the notation of [2].

Let $V$ be the subgroup of $G$ of elements having ones on the main diagonal and zeros above the main diagonal. Then $G$ acts on $V$ in a natural way; we denote the action of $a \in G$ on $v \in V$ by $v \Sigma$ (the transformations $v \mapsto v \Sigma$ are linear fractional transformations when $n = 2$ and generalizations thereof in higher dimensions). The operators of the representation $T(\cdot, \lambda)$ are given by

$$T(a, \lambda)f(v) = m(v, a; \lambda)f(v \Sigma)$$

where $m(v, a; \lambda)$ is an appropriate multiplier, and the underlying Hilbert space is $L_2(V)$.

In order to state our results we introduce a tube $\mathfrak{t}$ lying in the complex hyperplane $s_1 + s_2 + \cdots + s_n = 0$. The base $\mathcal{B}$ of $\mathfrak{t}$ is the smallest convex set which contains the points $(\sigma, -\sigma, 0, 0, \cdots, 0)$, $-1 < \sigma < 1$ and is invariant under all permutations of coordinates. A

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point \((s_1, s_2, \cdots, s_n)\), \(s_j = \sigma_j + it_j\) belongs to 3 if and only if \(s_1 + s_2 + \cdots + s_n = 0\) and \((\sigma_1, \sigma_2, \cdots, \sigma_n) \subseteq B\).

Theorem 1. For each character \(\lambda = \lambda(m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)\) for which \((s_1, s_2, \cdots, s_n) \subseteq 3\) there exists a representation \(a \rightarrow R(a, \lambda)\) of \(G\) on \(L_2(V)\). The family of all such representations has the following properties:

1. For each fixed \(\lambda\), \(a \rightarrow R(a, \lambda)\) is a continuous uniformly bounded representation.
2. For fixed \(a\), \(R(a, \lambda)\) as a function of \((s_1, s_2, \cdots, s_n)\) is analytic in the tube 3.
3. When \(\lambda\) is unitary, \(R(\cdot, \lambda)\) is unitarily equivalent to \(T(\cdot, \lambda)\), the corresponding member of the principal series.
4. \(R(\cdot, \lambda) = R(\cdot, \rho\lambda)\) for every "permutation" \(\rho\) of the character \(\lambda\).
5. \(R(\cdot, \lambda)' = R(\cdot, \lambda')\) where \(R(a, \lambda)' = R(a^{-1}, \lambda)^*\) and \(\lambda'(c) = \overline{\lambda(c)}\).

In their work on the principal series Gelfand and Neumark obtained a trace formula for certain operators associated with the representations \(a \rightarrow T(a, \lambda)\), [2, p. 73]. The formula involves a function (the character of the representation) which we denote by \(\psi(a, \lambda)\). It is initially defined for unitary characters \(\lambda\) but extends in an obvious way by analyticity to nonunitary characters. We prove the following result.

Theorem 2. Let \(\lambda\) be as in Theorem 1 and \(f\) be the convolution of two bounded functions on \(G\) with compact support. Then the operator

\[
R(f, \lambda) = \int_a f(a) R(a, \lambda) da
\]

is of trace class, and

\[
\text{tr} R(f, \lambda) = \int_a f(a) \psi(a, \lambda) da.
\]
(iii) By means of Theorem 2 we also show that if $\lambda' \neq p\lambda$ for all $p$, then the representation $R(\cdot, \lambda)$ is not equivalent to a unitary one, although it is uniformly bounded. The existence of a group $(SL(2, R))$ and uniformly bounded representations of it not equivalent to unitary ones was first proved by Ehrenpreis and Mautner [1].

We shall briefly describe the ideas behind the proofs of Theorems 1 and 2. We begin by limiting our attention to unitary characters $\lambda$. A basic fact we prove can be stated as follows:

**Lemma.** Let $G_0$ be the subgroup of $G$ consisting of the matrices $a$ such that $a_{jn} = 0$, $1 \leq j \leq n - 1$. For any character $\lambda$, call the integer $r$, uniquely determined by $0 \leq r < n$, $r = m_1 + m_2 + \cdots + m_n$, the residue of the character $\lambda$. Then if $\lambda_1$ and $\lambda_2$ have the same residue the restrictions of $T(\cdot, \lambda_1)$ and $T(\cdot, \lambda_2)$ to $G_0$ are unitarily equivalent.

Thus when we restrict to $G_0$, there are only $n$ (essentially) different representations among the principal series. This is remarkable in view of the known fact that the members of the principal series are all irreducible on $G_0$ [2, p. 22]. Hence there is a unitary operator $W(\lambda)$ so that if we set

$$R(a, \lambda) = W(\lambda)T(a, \lambda)W^{-1}(\lambda)$$

then $R(a, \lambda)$ for fixed $a \in G_0$ depends only on the residue of $\lambda$. We call the representations $R(\cdot, \lambda)$ the *normalized principal series*. It is these that can be continued analytically (i.e., to nonunitary $\lambda$'s) as in Theorem 1, while the $T(\cdot, \lambda)$ cannot.

The actual construction of the operators $W(\lambda)$ is too complicated to describe here but is intimately connected with the construction of the intertwining operators $A(p, \lambda)$ between $T(\cdot, \lambda)$ and $T(\cdot, p\lambda)$. In fact, it follows from part (4) of Theorem 1 that up to a constant multiple

$$A(p, \lambda) = W^{-1}(p\lambda)W(\lambda).$$

Moreover, we show that the operator $W(\lambda)$ can be written as a product of $n(n - 1)/2$ operators of the type $A(p, \lambda)$.

The work described above has many points of contact with the analysis of the special case of the $2 \times 2$ (real) group which we carried out previously in [4]. However, there is an essential difference. This is due to the fact that the study of the representations in question is closely related to the Fourier analysis on $L_2(V)$. When $n = 2$, $V$ is a commutative group; but this is not so when $n > 2$, and therein lies the major obstacle to the proof of Theorem 1.
REFERENCES


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