THE GENERAL COEFFICIENT THEOREM AND CERTAIN APPLICATIONS[1]

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Teichmüller was the first person to point out explicitly the connection between quadratic differentials and the solutions of certain extremal problems in Function Theory. He enunciated the principle that if a point is required to be fixed the quadratic differential will have a simple pole there, if in addition fixed values are required for the first n derivatives of competing functions the quadratic differential will have a pole of order $n+1$. He was led to this principle by abstraction from the numerous results of Grötzsch and by his considerations on quasiconformal mapping. However, he never proved any general result embodying this principle.

The General Coefficient Theorem provides such a result and includes as special cases virtually every result in the theory of univalent functions. We now formulate it in the following form [**6**; **8**], more general than that of earlier statements [**1**; **2**].

GENERAL COEFFICIENT THEOREM. *Let $\mathfrak{R}$ be a finite oriented Riemann surface, $Q(z)dz^2$ a positive quadratic differential on $\mathfrak{R}$, $\{\Delta\}$ an admissible family of domains $\Delta_j, j=1, \cdots, K$, on $\mathfrak{R}$ relative to $Q(z)dz^2$ and $\{f\}$ an admissible family of functions $f_j, j=1, \cdots, K$, associated with $\{\Delta\}$. Let $Q(z)dz^2$ have double poles $P_1, \cdots, P_r$ and poles $P_{r+1}, \cdots, P_n$ of order greater than two. We allow either of these sets to be void but not both. Let $P_j, j \leqq r$, lie in the domain Δ_l and in terms of a local parameter z representing P_j as the point at infinity let f_l have the expansion*

$$(1) \qquad f_l(z) = a^{(j)} z + a_0^{(j)} + \textit{negative powers of } z$$

and Q the expansion

$$(2) \qquad Q(z) = \alpha^{(j)} z^{-2} + \textit{higher powers of } z^{-1}.$$

Let P_j, $j>r$, a pole of order m_j greater than two lie in the domain Δ_l and in terms of a local parameter z representing P_j as the point at infinity let f_l have the expansion

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$$f_l(z) = z + \sum_{i=k_j}^{\infty} a_i^{(j)}/z^i \tag{3}$$

where k_j is the smallest integer greater than or equal to $(1/2)m_j - 2$ and Q the expansion

$$Q(z) = \alpha^{(j)}\left[z^{m_j-4} + \sum_{i=k_j+1}^{\infty} \beta_i^{(j)} z^{m_j-i-4}\right]. \tag{4}$$

Then

$$\Re\left\{\sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)} + \sum_{=r+1}^{n} \alpha^{(j)}\left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2}m_j - 2\right)\epsilon_j\left(a_{k_j}^{(j)}\right)^2 + \epsilon_j\beta_{k_j+1}^{(j)} a_{k_j}^{(j)}\right]\right\} \leqq 0, \tag{5}$$

where $\log a^{(j)} = \log|a^{(j)}| - id(F, P_j)$, $j \leqq r$ *and* $\epsilon_j = 1$ *if* m_j *is even*; $\epsilon_j = 0$ *if* m_j *is odd*, $j > r$.

If equality occurs in (5) *each* f_j, $j = 1, \cdots, K$, *must be an isometry in the Q-metric*

$$|d\zeta| = |Q(z)|^{1/2}|dz|,$$

each trajectory of $Q(z)dz^2$ *in* $\bigcup_{j=1}^{K}\Delta_j$ *must go into another such and the set* $\bigcup_{j=1}^{K} f_j(\Delta_j)$ *must be dense in* $\Re$. *If equality occurs in* (5) f_l *reduces to the identity in a domain* Δ_l *for which any of the following conditions holds.*

(i) *There is in* Δ_l *a pole* P_j, $j > r$, *of order* m_j *such that* $a_i^{(j)} = 0$ *for* $i < \min(k_j + 1, m_j - 3)$.

(ii) *There is in* Δ_l *a pole* P_j, $j \leqq r$, *with the corresponding coefficient* $a^{(j)}$ *equal to one.*

(iii) *There is in* Δ_l *a simple pole of* $Q(z)dz^2$ *or a point on a trajectory ending in a simple pole.*

Equality can occur in (5) *when there exists a double pole* P_j, $j \leqq r$, *such that for the corresponding coefficient* $|a^{(j)}| \neq 1$ *only when* $\Re$ *is conformally equivalent to the sphere and* $Q(z)dz^2$ *is a quadratic differential whose only critical points are two poles each of order two. If further* $\{\Delta\}$ *consists of a single domain the corresponding admissible function is conformally equivalent to a linear transformation with the points corresponding to these poles as fixed points.*

By a quadratic differential on a Riemann surface $\Re$ we mean an entity which assigns to every local uniformizing parameter z of $\Re$ a function $Q(z)$ meromorphic in the neighborhood for z and satisfying the following condition. If z^* is a second local uniformizing parameter of $\mathfrak{R}$, if $Q^*(z^*)$ is the function associated with z^* and if the neighborhood on $\mathfrak{R}$ for z^* overlaps that for z, then at common points of these neighborhoods we have

$$Q^*(z^*) = Q(z)\left(\frac{dz}{dz^*}\right)^2.$$

We denote quadratic differentials generically by symbols such as $Q(z)dz^2$. Clearly we may speak of a quadratic differential having zeros and poles of specified order. We denote the set of zeros and simple poles of $Q(z)dz^2$ by C, the set of other poles of $Q(z)dz^2$ by H. A maximal regular curve on which $Q(z)dz^2>0$ is called a trajectory of the quadratic differential, one on which $Q(z)dz^2<0$ an orthogonal trajectory. These curves are evidently independent of the choice of local uniformizing parameters.

In a neighborhood of a point of $\mathfrak{R}$ which is neither a zero nor a pole of $Q(z)dz^2$ the trajectories behave like a regular curve family. A zero of order μ or a simple pole ($\mu=-1$) is the limiting end point of $\mu+2$ trajectory arcs equally spaced at angles of $2\pi/(\mu+2)$. At a pole of order two the trajectories behave locally either like radial arcs, logarithmic spirals or concentric circles. At a pole of order $\mu(\mu>2)$ there are $\mu-2$ asymptotic directions for trajectories, equally spaced at angles of $2\pi/(\mu-2)$. More detailed descriptions of these various situations will be found in [**2**, §3.2].

If $\mathfrak{R}$ is a finite oriented Riemann surface, a positive quadratic differential on $\mathfrak{R}$ is a quadratic differential on $\mathfrak{R}$ such that $Q(z)dz^2$ is regular at boundary points and $Q(z)dz^2\geqq 0$ in terms of boundary uniformizers. Perhaps the most essential step in the proof of the General Coefficient Theorem is the analysis of the global structure of the trajectories of such a positive quadratic differential. The answer is contained in the Basic Structure Theorem. The present form of this result is given in [**7**].

BASIC STRUCTURE THEOREM. *Let $\mathfrak{R}$ be a finite oriented Riemann surface and $Q(z)dz^2$ a positive quadratic differential on $\mathfrak{R}$ where we exclude the following possibilities and all configurations obtained from them by conformal equivalence*:

I. *$\mathfrak{R}$ the z-sphere, $Q(z)dz^2=dz^2$,*

II. *$\mathfrak{R}$ the z-sphere, $Q(z)dz^2=Ke^{i\alpha}dz^2/z^2$, α real, K positive.*

III. *$\mathfrak{R}$ a torus, $Q(z)dz^2$ regular on $\mathfrak{R}$,*

Let Λ denote the union of all trajectories of $Q(z)dz^2$ which have one limiting end point at a point of C and a second limiting end point at a point of $C\cup H$. Then

(i) $\mathfrak{R}-\bar{\Lambda}$ *consists of a finite number of end, strip, ring, circle and density domains*;

(ii) *each such domain is bounded by a finite number of trajectories together with the points at which the latter meet*; *every boundary component of such a domain contains a point of* C, *except that a boundary component of a circle or ring domain may coincide with a boundary component of* $\mathfrak{R}$; *for a strip domain the two boundary elements arising from points of* H *divide the boundary into two parts on each of which is a point of* C;

(iii) *every pole of* $Q(z)dz^2$ *of order* m *greater than two has a neighborhood covered by the inner closure of the union of* $m-2$ *end domains and a finite number* (*possibly zero*) *of strip domains*;

(iv) *every pole of* $Q(z)dz^2$ *of order two has a neighborhood covered by the inner closure of the union of a finite number of strip domains or has a neighborhood contained in a circle domain.*

End and strip domains are mapped respectively on half-planes and horizontal strips by $\int(Q(z))^{1/2}dz$. Circle and ring domains are mapped respectively by $\exp(k\int(Q(z))^{1/2}dz)$ for suitable constants k on circles and circular rings. Density domains are swept out (apart from points of C) by trajectories each of which is everywhere dense in the domain.

An admissible family of domains $\{\Delta\}$ relative to a positive quadratic differential $Q(z)dz^2$ on a finite oriented Riemann surface $\mathfrak{R}$ is obtained by slitting $\mathfrak{R}$ along a finite number of trajectories of $Q(z)dz^2$ which either are closed or join two points of C and along a finite number of arcs on closures of trajectories in $\mathfrak{R}-H$. An admissible family of functions $\{f\}$ associated with $\{\Delta\}$ then consists of a family of conformal mappings of the domains Δ_j comprising $\{\Delta\}$ onto nonoverlapping domains in $\mathfrak{R}$, leaving fixed all poles of $Q(z)dz^2$ interior to these domains, normalized by the expansions (3) at poles of order greater than two and admitting an admissible homotopy F into the identity [**2**, p. 49] such that for a pole of order greater than two on the boundary of a strip domain the deformation degree of F at that pole is zero. As usual $d(F, P_j)$ denotes the deformation degree of F at the pole P_j [**2**, p. 50].

In the statement of the General Coefficient Theorem the requirement that certain initial coefficients in the development (3) should vanish is necessary for the application of the technique used in the proof of the result. However, the corresponding restriction on the coefficients in the development (4) of the quadratic differential is made only to achieve formal simplicity in the expression (5) and can always be attained by a suitable choice of the local parameters used at the various poles in question.

In the proof proper of the General Coefficient Theorem the first step is to show that it is sufficient to proceed on the assumption that $Q(z)dz^2$ has no simple poles on $\mathfrak{R}$. This is done by replacing $\mathfrak{R}$ by a suitable covering surface. Then the double of $\mathfrak{R}-H$ becomes a complete differential geometric surface with the metric induced by the Q-metric.

The main part of the proof consists of drawing suitable curves around each point of H and regarding the set obtained by deleting these curves and their interiors from $\bigcup_{j=1}^{K} \Delta_j$. Then the areas of this set and the corresponding portion of $\bigcup_{j=1}^{K} f_j(\Delta_j)$ in the Q-metric are compared in two ways. On the one hand we obtain an inequality in one direction by studying the behaviour of the functions f_j on the above curves. An inequality in the opposite direction is obtained by applying the method of the extremal metric in each of the various domains associated with the trajectory structure of $Q(z)dz^2$. Combining these inequalities we obtain, after formal reduction, the inequality (5).

The equality statements are proved by applying the usual equality treatment associated with the method of the extremal metric.

The applications of the General Coefficient Theorem include all the standard results in the theory of univalent functions and these are presented in [2]. As an illustration we give here a simple one to show the working of the method.

Let $f \in S$. Then

$$\frac{r}{(1+r)^2} \leqq |f(z)| \leqq \frac{r}{(1-r)^2}, \qquad |z| = r,\ 0 < r < 1,$$

equality occurring on each side only for $f(z) = z(1+e^{-i\theta}z)^{-2}$, θ real, respectively for $z = re^{i\theta}$ and $-re^{i\theta}$.

In proving the lower bound we let $\tilde{f}(z) = z(1+z)^{-2}$, $\Delta = \tilde{f}(E)(E\colon |z| < 1)$, $\Phi(w)$ be the inverse of $\tilde{f}$ on Δ, $d = r(1+r)^{-2}$ and $c = f(re^{i\theta})$, θ real. We take $\mathfrak{R}$ to be the w-sphere, the quadratic differential

$$Q(w)dw^2 = \frac{dw^2}{w^2(w-d)},$$

the admissible domain Δ given above and

$$g(w) = \frac{d}{c} f(e^{i\theta}\Phi(w))$$

an admissible function associated with Δ.

The quadratic differential has a double pole P_1 at the origin. The corresponding coefficients are

$$\alpha^{(1)} = -d^{-1}, \qquad a^{(1)} = c/de^{i\theta}.$$

In this case the inequality (5) becomes

$$\Re\{-d^{-1}\log(c/de^{i\theta})\} \leqq 0$$

that is

$$|f(re^{i\theta})| \geqq \frac{r}{(1+r)^2}.$$

In order for equality to occur by equality condition (iii) we must have

$$\frac{c}{d} f(e^{i\theta}\Phi(w)) \equiv w;$$

then $c = de^{i\theta}$ and setting $\Phi(w) = e^{-i\theta}z$ we have $f(z) = z(1+e^{-i\theta}z)^{-2}$.

In proving the upper bound we let $\tilde{f}$, Δ, Φ and $\Re$ be as above. We now set $d = -r(1-r)^{-2}$, $c = f(re^{i\theta})$,

$$Q(w)dw^2 = \frac{dw^2}{w^2(w-d)},$$

$$g(w) = \frac{d}{c} f(-e^{i\theta}\Phi(w))$$

the latter being an admissible function associated with Δ.

The quadratic differential has a double pole P_1 at the origin. The corresponding coefficients are

$$\alpha^{(1)} = -d^{-1}, \qquad a^{(1)} = -c/de^{i\theta}.$$

In this case the inequality (5) becomes

$$\Re\{-d^{-1}\log(-c/de^{i\theta})\} \leqq 0$$

that is

$$|f(re^{i\theta})| \leqq \frac{r}{(1-r)^2}.$$

The same argument as before shows that equality can occur here only for the function $f(z) = z(1-e^{-i\theta}z)^{-2}$.

The other applications of the General Coefficient Theorem given in [2] may be classified essentially as region of values results and theorems for families of univalent functions. Since the publication of that work numerous other applications have been made. In [3] explicit formulas are given for the solution of problems involving weighted distortion in conformal mapping. Univalent functions with real coefficients are treated with considerable completeness in [4]. In particular, imposing a suitable normalization, there are determined the exact region covered by the image of the unit circle under every such function, the region of values of such functions at a point in the unit circle and bounds involving the derivatives of these functions. These ideas have also been used in a new and unified approach to treating the initial coefficients of normalized univalent functions defined either on the interior or on the exterior of the unit circle [5]. In this manner both known and new inequalities are obtained. In [9] the same principles are applied to normalized univalent functions for which certain initial coefficients vanish. For these the earlier results are extended to apply to coefficients of any index. Finally in [6] are to be found more sophisticated applications to families of univalent functions.

We will indicate here in some detail one other application because it presents a somewhat different aspect. It is the use of the General Coefficient Theorem to find a new lower bound for the schlicht Bloch constant $\mathfrak{A}$. This constant is defined by the property that every function in S provides a mapping of $|z|<1$ onto a domain which contains some open circle of radius $\mathfrak{A}$ while this is not so for any larger constant. Landau proved by elementary means that in finding lower bounds for $\mathfrak{A}$ we may restrict ourselves to functions satisfying $(1-|z|^2)|f'(z)|\leqq 1$. This inequality implies the conditions

$$A_2 = 0,$$

$$|A_3| \leqq \frac{1}{3},$$

$$|f(z)| \leqq \frac{1}{2}\log\frac{1+|z|}{1-|z|},$$

where $f(z)$ is to have about the origin the expansion

$$f(z) = z + A_2z^2 + A_3z^3 + \cdots .$$

Using the first two conditions Landau obtained a lower bound for $\mathfrak{A}$ which is the best which can be obtained on the basis of this information alone.

Recently Reich obtained an improved lower bound using in addition the third condition. His essential step is to observe that the function

$$f(z, t) = \frac{1}{t} f(tz), \qquad 0 < t < 1,$$

lies in S and satisfies the bound

$$|f(z, t)| \leqq M(t), \qquad |z| < 1,$$

where

$$M(t) = \frac{1}{2t} \log \frac{1+t}{1-t}$$

and has the expansion

$$f(z, t) = z + t^2 A_3 z^3 + \cdots .$$

Let

$$Q(w, M, p)dw^2 = -\frac{(w - M)^2(w^4 + \cdots + 1)}{w^4(w - p)(w - p^{-1})} dw^2$$

be a quadratic differential with real coefficients which is a positive quadratic differential on $|w| < M$ where M is a parameter susceptible of all real values greater than one, $0 < p < M$, and $Q(w, M, p)dw^2 \geqq 0$ on the segment $p < w < M$. Further let $f(z, M, p)$ lie in S and map $|z| < 1$ onto an admissible domain with respect to $Q(w, M, p)dw^2$ which does not contain any point of the segment $p < w < M$. Let $f(z, M, p)$ have the expansion

$$f(z, M, p) = z + A_3(M, p)z^3 + \cdots .$$

Now suppose $f(z)$ omits the value γ in $|z| < 1$. Then $f(z, t)$ omits the value γt^{-1}. If the value p can occur above for the choice $M = M(t)$ and we had

$$|\gamma t^{-1}| \leqq p,$$

then for suitable real θ, $e^{-i\theta}f(e^{i\theta}z, t)$ would map $|z| < 1$ onto a domain omitting p. Then by a direct application of the General Coefficient Theorem we would obtain

$$\Re\{t^2 e^{2i\theta} A_3\} < A_3(M(t), p),$$

the strict inequality being assured by a consideration of the equality conditions. This would imply

$$-\frac{1}{3} t^2 < A_3(M(t), p).$$

Thus if it is possible to solve the equation

$$-\frac{1}{3}t^2 = A_3(M(t), p)$$

for $p=p(t)$ satisfying the above conditions we must have

$$|\gamma t^{-1}| > p(t)$$

or

$$|\gamma| > tp(t).$$

For every admissible value of t, $tp(t)$ then provides a lower bound for $\mathfrak{A}$ and t may be chosen to make this bound most advantageous.

This procedure is carried out with explicitly given mappings in the paper [**10**] to obtain the numerical bound $\mathfrak{A}>.5705$. There is indicated also the possibility of various small improvements.

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