The purpose of the present note is to sketch a solution for the problem of determining the form of all isometries of any reflexive Orlicz space.¹ A partial result in that direction was obtained earlier by J. Lamperti [4] (who suggested this problem to us recently). The ideas of the proof are very closely related to those used recently by the author to develop a unified and slightly extended theory (unpublished) [6] for the classical results of Banach [1], Stone [8] and Kadison [2] (see also [4]) on isometries of $C(X)$, $L_p$ spaces, and $C^*$ algebras. The systematic use of semi-inner-product spaces, and generalized hermitians [5], plays a central role. A semi-inner-product space, is a vector space $X$ on which there is defined a (complex valued) form $[x, y]$ satisfying:

(i) Linearity in $x$,
(ii) $[x, x] > 0$ if $x \neq 0$,
(iii) $|[x, y]|^2 \leq [x, x][y, y]$.

$X$ is then normed under $\|x\| = [x, x]^{1/2}$.

From now on, $X$ is a reflexive Orlicz space [7; 3] whose unit sphere is the set \{ $f \in X$: $\int \phi(|f|) \leq 1$ \}. It is somewhat laborious but not very difficult to show that the semi-inner-product for $X$ is given by:

$$[f, g] = C(g) \int f \phi\left( \left| \frac{g}{\|g\|} \right| \right) \operatorname{sgn} g$$

where

$$\operatorname{sgn} g = \begin{cases} \left| \frac{g}{\|g\|} \right| & \text{if } g \neq 0, \\ g & \text{if } g = 0 \end{cases}$$

with $C(g) = \left( \int g \phi\left( \left| \frac{g}{\|g\|} \right| \right) \operatorname{sgn} g \right)^{-1} |g|^2$, when $g$ is such that the measure of \{ $\xi \in \Omega$: $\phi$ has no derivative at the point $|g(\xi)|/\|g\|$ \} is 0.

A bounded hermitian operator (see [5]) satisfies by definition $[Hf, f] = \text{real for all } f \in X$.

**Proposition 1.** If $h$ is real valued and in $L_\infty(\Omega)$, $Hf = hf$ defines a hermitian operator on $X$, and $\|H\| = \|h\|_\infty$. ¹ Actually the proof sketched below covers the Orlicz spaces over measure spaces containing no atoms. If the measure space contains atoms, further argument is needed.
THEOREM 2. If $X$ is different from $L_2(\Omega)$, $H$ is a bounded hermitian on $X$, then there is a real valued $h \in L_\infty(\Omega)$ such that $Hf = hf$ for all $f \in X$, and $\|H\| = \|h\|_\infty$.

SKETCH OF THE PROOF. If $u$ and $v$ are in $X$, and have disjoint supports, $\Omega_1$ and $\Omega_2$, then $\text{Im} \left[ H(e^{i\alpha u} + e^{i\beta v}), e^{i\alpha u} + e^{i\beta v} \right] = 0$. $\alpha, \beta$ real and arbitrary lead to.

$$\int_{\Omega_2} H u \phi'(\frac{|v|}{\|v\|}) \text{ sgn } v = \left\{ \int_{\Omega_1} H v \phi'(\frac{|u|}{\|u\|}) \text{ sgn } u \right\}.$$ 

One applies this to $u_2 = \alpha \chi_{\Omega_1}, u_3 = \beta \chi_{\Omega_1}, u_1 = (\alpha + \beta) \chi_{\Omega_1}$ and $v = \chi_{\Omega_2}/\|\chi_{\Omega_2}\|$, where $\chi_\Omega$ denotes the characteristic function of the measurable set $\Omega$. One arrives finally at:

$$\left[ \phi'\left(\frac{\alpha + \beta}{\lambda_1}\right) - \frac{\phi'\left(\frac{1}{\lambda_1}\right)}{\phi'\left(\frac{1}{\lambda_2}\right)} \phi'\left(\frac{\alpha}{\lambda_2}\right) - \frac{\phi'\left(\frac{1}{\lambda_1}\right)}{\phi'\left(\frac{1}{\lambda_3}\right)} \phi'\left(\frac{\beta}{\lambda_3}\right) \right] \int_{\Omega_1} H v = 0$$

where $\lambda_1 = \|u_1 + u_2 + v\|, \lambda_2 = \|u_1 + v\|, \lambda_3 = \|u_2 + v\|$, $\alpha, \beta > 0$ arbitrary $\Omega_1, \Omega_2$ and $v$ fixed. Letting the measure of $\Omega_1$ tend to 0 in a convenient manner $\lambda_1, \lambda_2$ and $\lambda_3$ tend to $\|v\| = 1$, so that either $\phi'(\alpha + \beta) = \phi'(\alpha) + \phi'(\beta)$ (i.e., $\phi(\alpha) = k\alpha^2$ and $X$ is $L_2(\Omega)$) or else $Hv$ is 0 on $\Omega_3$. From this follows that if $f \in X$ is a step function and $\Omega_0$ the support of one step, $H(f - f(\Omega_0) 1)$ is 0 on $\Omega_0$, hence $Hf = hf$, where $h = H1$. The rest is immediate. From this we obtain the main theorem.

THEOREM 3. If $U$ is an isometry from $X$ onto $X$, then it is of the form $Uf(\cdot) = u(\cdot)f(T\cdot)$ where $T$ is a measurable transformation in $\Omega$ and $u$ a fixed function in $X$, unless $X$ is a Hilbert space.

SKETCH OF THE PROOF. The expression $[f, g]' = [Uf, Ug]$ is again a semi-inner-product on $X$, so that if $H$ is hermitian the same holds for $UHU^{-1}$. If the real-valued function $h \in L_\infty(\Omega)$, denote by $H_h$ the multiplication operation defined by $h$ (which is hermitian). $UH_h U^{-1} = H_h$, where $\|H\| = \|H\|_\infty$. Since $UH_h U^{-1} UH_h U^{-1} = UH_h U^{-1}$, the operation $\cdot$ is multiplicative, and step functions go into step functions. This defines $T$; the rest goes smoothly.

REMARK. The previous argument could be modified so as to hold for a form not satisfying condition (iii), if a sufficiently strong cond-

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\*\* From a letter I received recently from Dr. C. A. McCarthy, it appears that McCarthy had a proof of Theorem 2.
tion is assumed with respect to $\phi$. The space would not be an Orlicz space, but an extension of the $L_p$ space for $p < 1$. For the latter $L_p$ spaces, it is known that the isometries are as described above.

REFERENCES

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ON THE RECURRENCE OF SUMS OF RANDOM VARIABLES

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We give a very short proof of the recurrence theorem of Chung and Fuchs \cite{1} in one and two dimensions. This new elementary proof does not detract from the old one which uses a systematic method based on the characteristic function and yields a satisfactory general criterion. But the present method, besides its brevity, also throws light on the combinatorial structure of the problem.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{M}$ that of positive real numbers. Let \{\(X_n, n \in \mathbb{N}\}\} be a sequence of independent, identically distributed real-valued random vectors, and let \(S_n = \sum_{r=1}^n X_r\). The value $x$ is possible iff for every $\varepsilon > 0$ there exists an $n$ such that $P\{|S_n - x| < \varepsilon\} > 0$; it is recurrent iff for every $\varepsilon > 0$, $P\{|S_n - x| < \varepsilon\}$ for infinitely many $n$ = 1. It is shown in \cite{1} that every possible value is recurrent if and only if for some $m \in \mathbb{M}$ we have

$$\sum_{n=1}^{\infty} P\{|S_n| < m\} = \infty.$$  

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