VECTOR FIELDS ON SPHERES

BY J. F. ADAMS

Communicated by Deane Montgomery, October 9, 1961

Let us write \( n = (2a + 1)2^b \), where \( a \) and \( b \) are integers, and let us set \( b = c + 4d \), where \( c \) and \( d \) are integers and \( 0 \leq c \leq 3 \); let us define \( \rho(n) = 2^c + 8d \). Then it follows from the Hurwitz-Radon-Eckmann theorem in linear algebra that there exist \( \rho(n) - 1 \) vector fields on \( S^{n-1} \) which are linearly independent at each point of \( S^{n-1} \) (cf. [4]).

**Theorem 1.1.** With the above notation, there do not exist \( \rho(n) \) linearly independent vector fields on \( S^{n-1} \).

This theorem asserts that the known positive result, stated above, is best possible. Like the theorems given below, it is copied without change of numbering from a longer paper now in preparation.

Theorem 1.1 may be deduced from the following result (cf. [1]).

**Theorem 1.2.** The truncated projective space \( RP^{m+p(m)}/RP^{m-1} \) is not coreducible; that is, there is no map \( f: RP^{m+p(m)}/RP^{m-1} \to S^m \) such that the composite

\[
S^m = RP^m/RP^{m-1} \xrightarrow{i} RP^{m+p(m)}/RP^{m-1} \xrightarrow{f} S^m
\]

has degree 1.

Theorem 1.2 is proved by employing the "extraordinary cohomology theory" \( K(X) \) of Atiyah and Hirzebruch [2; 3]. If our truncated projective space \( X \) were coreducible, then the group \( K(X) \) would split as a direct sum, and this splitting would be compatible with any "cohomology operations" that one might define in the "cohomology theory" \( K(X) \).

**Theorem 5.1.** It is possible to define operations

\[
\Psi^k_A: K_A(X) \to K_A(X)
\]

for any integer \( k \) (positive, negative or zero) and for \( A = R \) (real numbers) or \( A = C \) (complex numbers). These operations have the following properties.

(i) \( \Psi^k_A \) is natural for maps of \( X \).
(ii) \( \Psi^k_A \) is a homomorphism of rings with unit.
(iii) The following diagram is commutative.

---

\(^1\) Supported in part by the National Science Foundation under grant G14779.
(Here the homomorphism $c$ is induced by "complexification" of real bundles.)

(iv) $\Psi^k_A \Psi^k_A = \Psi^k_A$.

(v) $\Psi^k_A$ and $\Psi^{-1}_R$ are identity functions. $\Psi^k_R$ assigns to each bundle over $X$ the trivial bundle with fibres of the same dimension. $\Psi^k_C$ assigns to each complex bundle over $X$ the "complex conjugate" bundle.

(vi) If $\xi \in K_C(X)$ and $\text{ch}_C(\xi)$ denotes the $2q$-dimensional component of the Chern character $\text{ch}_C(\xi)$, then

$$\text{ch}^q(\Psi^k_C \xi) = k^q \text{ch}^q(\xi).$$

This theorem is proved using virtual representations of groups. By (iv), (v) it is sufficient to define $\Psi^k_A$ for $k>0$. One can define polynomials $Q^k_n$ by setting

$$(x_1)^k + (x_2)^k + \cdots + (x_n)^k = Q^k_n(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

where $\sigma_i$ is the $i$th elementary symmetric function of $x_1, x_2, \cdots, x_n$. One can define a virtual representation of $GL(n, A)$ by setting

$$\psi^k_A = Q^k_n(E^1_A, E^2_A, \cdots, E^n_A)$$

where $E^i_A$ denotes the $i$th exterior power representation. The operations $\Psi^k_A$ are induced by the virtual representations $\psi^k_A$.

The values of our groups $K(X)$ and of our operations in them are given by the following result. In order to state it, we define $\phi(m, n)$ to be the number of integers $s$ such that $m < s \leq n$ and $s \equiv 0, 1, 2$ or $4$ mod $8$.

**Theorem 7.4.** Assume $m \neq -1 \mod 4$. Then $\tilde{K}_R(RP^n/RP^m) = \mathbb{Z}/4$, where $f = \phi(m, n)$. If $m = 0$ then the canonical real line-bundle $\xi$ yields a generator $\lambda = \xi - 1$, and the polynomials in $\lambda$ are given by the formula

$$\lambda Q(\lambda) = Q(-2)\lambda,$$

where $Q$ is any polynomial with integer coefficients. Otherwise the projection $RP^n \to RP^n/RP^m$ maps $\tilde{K}_B(RP^n/RP^m)$ isomorphically onto the subgroup of $\tilde{K}_R(RP^n)$ generated by $\lambda^{g+1}$, where $g = \phi(m, 0)$. We write $\lambda^{g+1}$ for the element in $\tilde{K}_B(RP^n/RP^m)$ which maps into $\lambda^{g+1}$.
In the case \( m = -1 \mod 4 \) we have
\[
\tilde{K}_R(\mathbb{RP}^n/\mathbb{RP}^{4t-1}) = \tilde{K}_R(\mathbb{RP}^n/\mathbb{RP}^{4t}) + Z;
\]
here the first summand is embedded by an induced homomorphism and the second is generated by a suitable element \( \tilde{\lambda}^{(2)} \), where \( g = \phi(4t, 0) \).

The operations are given by the following formulae.

\[(i) \quad \Psi_R^k \lambda^{(2)} = \begin{cases} 0 & \text{ (k even)}, \\ \lambda^{(2k+1)} & \text{ (k odd)}; \end{cases}\]

\[(ii) \quad \Psi_R^k \lambda^{(2)} = k^{2t-2} \lambda^{(2)} + \begin{cases} \frac{1}{2} k^{2t} \lambda^{(2k+1)} & \text{ (k even)}, \\ \frac{1}{2} (k^{2t} - 1) \lambda^{(2k+1)} & \text{ (k odd)}. \end{cases}\]

This theorem is proved by deducing results in the following order:
(i) Results on complex projective spaces for \( \Lambda = \mathbb{C} \).
(ii) Results on real projective spaces for \( \Lambda = \mathbb{C} \).
(iii) Results on real projective spaces for \( \Lambda = \mathbb{R} \).

References


The Institute for Advanced Study