ON THE DEGREE OF DIFFERENTIABILITY OF CURVES USED IN THE DEFINITION OF THE HOLONOMY GROUP

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Let $P(M, G)$ be a principal fiber bundle with Lie structure group G and base M, with differentiability of class C^∞ for all manifolds and mappings involved in its definition. Given a connection of class C^∞ in P, the holonomy group $\Phi(x)$ with reference point $x \in P$ is defined as the subgroup of G consisting of all elements $a \in G$ such that x and $x \cdot a$ can be joined by a horizontal curve in P. Here, a horizontal curve usually means either a piecewise C^1-curve with horizontal tangent vectors or a piecewise C^∞-curve with horizontal tangent vectors. In fact, for any specified degree of differentiability k, $1 \leqq k \leqq \infty$, we can use piecewise C^k-curves exclusively thus defining the holonomy group $\Phi_k(x)$ which depends presumably on k. Of course, it is evident that $\Phi_1(x) \supset \Phi_2(x) \supset \cdots \supset \Phi_\infty(x)$.

We asked ourselves whether these holonomy groups are the same or not, and, as far as we know, there has not been any record concerning this question. One might think that approximation of piecewise C^1-curves by piecewise C^∞-curves will settle this question, but this method does not seem to work too easily in view of the fact that a Lie group can admit a Lie subgroup of lower dimension which is everywhere dense.

We present here a proof of

THEOREM. $\Phi_1(x) = \Phi_\infty(x)$.

We follow the terminologies of [2] in which piecewise C^∞-curves are used exclusively. Let $P(x)$ be the set of all points in P which can be joined to x by a horizontal piecewise C^∞-curve. It is known that $P(x)$ is a subbundle of P with structure group $\Phi_\infty(x)$ [2, p. 37]. We define a distribution Δ on P by $\Delta_x = T_x(P(x))$ for each $x \in P$. We can prove that Δ is differentiable. It is involutive, since for each point x of P, $P(x)$ is an integral manifold of Δ through x. It is also easy to show that $P(x)$ is indeed a maximal integral manifold of Δ through x. (Remark that the distribution Δ considered in [2, p. 39] is the restriction of our distribution Δ here to $P(x)$ for a fixed x.) Now let $a \in \Phi_1(x)$. This means that there is a horizontal piecewise C^1-curve $x(t)$ such that

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$x(0)=x$ and $x(1)=x\cdot a$. From the lemma below, it follows that the curve $x(t)$ lies entirely in $P(x)$ so that $x\cdot a\in P(x)$. But every point of $P(x)$ which lies in the same fiber as x is of the form $x\cdot b$ with some $b\in\Phi_\infty(x)$. Thus $x\cdot a=x\cdot b$ and hence $a=b$, proving that $a\in\Phi_\infty(x)$.

LEMMA. *Let Δ be an involutive C^∞-distribution on a C^∞-manifold. Suppose $x(t)$, $0\leqq t\leqq 1$, is a piecewise C^1-curve whose tangent vectors (where they exist) belong to Δ. If $x(0)$ is in a maximal integral manifold W of Δ, then the curve $x(t)$ lies entirely in W.*

PROOF. We may assume that $x(t)$ is a C^1-curve. Take a local coordinate system $(x^1, \cdots, x^n)$ around the point $x(0)$ such that $\partial/\partial x^1, \cdots, \partial/\partial x^r$, $r=\dim\Delta$, form a local basis for Δ [1, p. 92]. For small values of t, say, $0\leqq t<\epsilon$, $x(t)$ can be expressed by $x^i=x^i(t)$, $1\leqq i\leqq n$, and its tangent vectors are given by $\sum_{i=1}^n (dx^i/dt)(\partial/\partial x^i)$. By assumption, we have $dx^i/dt=0$ for $r+1\leqq i\leqq n$. Thus $x^i(t)=x^i(0)$ for $r+1\leqq i\leqq n$, so that $x(t)$, $0\leqq t<\epsilon$, lies in the slice through $x(0)$ and hence in W. Now by the standard continuation argument, we see that the entire curve $x(t)$, $0\leqq t\leqq 1$, lies in W.

COROLLARY. *The restricted holonomy groups $\Phi_1^0(x)$ and $\Phi_\infty^0(x)$ coincide with each other.*

The restricted holonomy group $\Phi_k^0(x)$ is the subgroup of $\Phi_k(x)$ consisting of all elements $a\in G$ such that x and $x\cdot a$ can be joined by a horizontal piecewise C^k-curve in P whose projection on M is (continuously) homotopic to 0. It is known that for $k=1$ and ∞ [2, p. 32] $\Phi_k^0(x)$ is nothing but the arcwise component of the identity of the group $\Phi_k(x)$ (more precisely, $\Phi_k^0(x)$ is the set of all elements of $\Phi_k(x)$ which can be joined by a continuous curve in G which lies entirely in $\Phi_k(x)$). Since $\Phi_1(x)=\Phi_\infty(x)$, it is then clear that $\Phi_1^0(x)=\Phi_\infty^0(x)$.

REMARK. In the case where $P(M, G)$ is a real analytic bundle with an analytic connection, we can still define the holonomy group $\Phi_\omega(x)$ by using only piecewise analytic curves. Since we can develop the results used from [2] in the above discussion by using piecewise analytic curves only, it is clear that $\Phi_\omega(x)=\Phi_1(x)$ and $\Phi_\omega^0(x)=\Phi_1^0(x)$.

BIBLIOGRAPHY

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