SIMPLE NEIGHBORHOODS

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The purpose of this note is to state a collection of results, all related to questions of uniqueness of smooth neighborhoods of finite complexes as imbedded (nicely) in differentiable manifolds. The reported results will appear in subsequent papers, the first of which is [1]. The approach taken is always via the theory of simple homotopy types (due to J. H. C. Whitehead), and the main theorem below (Uniqueness of simple neighborhoods) is an application of the Nonstable Neighborhood Theorem, proved in [1] (suggested by the remarkable work of Smale on high-dimensional manifolds).

What is necessary in Differential Topology is:

(I) A theorem asserting the existence of a smooth neighborhood about any finite complex, nicely imbedded in a differentiable manifold, where smooth neighborhood is taken in the strictest possible sense.

(II) A theorem asserting uniqueness of smooth neighborhoods about any fixed nicely imbedded complex, where smooth neighborhood is taken in the weakest conceivable sense.

An existence theorem in the style of (I) is proved in Chapter 7 of [1]. The term neighborhood is as given in [1]. To fulfill the ambitions of (II), the concept of simple neighborhood is introduced (Definition 2), and the Uniqueness of Simple Neighborhoods Theorem is proved.

A strengthening of the notion of h-cobordism (more suitable to the framework of this theory) is s-cobordism (Definition 3, below), and an application of the Simple Neighborhood Uniqueness theorem is the result that any s-cobordism (between manifolds of dimension greater than 4) is trivial. Hence if two such manifolds are s-cobordant, then they are differentiably isomorphic. This is a generalization to non-simply-connected manifolds of the h-cobordism theorem of Smale. See [3].

In subsequent papers this theory will be applied to the study of differentiable knots, and the setting up of an obstruction theory of the type hinted about in [2].

A differentiable isomorphism $\phi: (A, B) \rightarrow (A', B')$ for differentiable manifolds, $A \supseteq B$, $A' \supseteq B'$ will mean an isomorphism $\phi: A \rightarrow A'$ such that

$$\phi \mid B: B \overset{\rightarrow}{\approx} B'$$

is an isomorphism of B onto B' if ϕ_0: $(A, B) \rightarrow (A', B')$, ϕ_1: $(A, B) \rightarrow (A', B')$ are isomorphisms then an isotopy ϕ_t: $(A, B) \rightarrow (A', B')$ will mean an isotopy in the usual sense ϕ_t: $A \rightarrow A'$ such that

$$\phi_t \,|\, (B) : B \overset{\rightarrow}{\approx} B'$$

is an isomorphism for $0 \leqq t \leqq 1$. If ϕ_0 is isotopic to ϕ_1, I denote that by:

$$\phi_0 \approx \phi_1.$$

Definition 1. Let K be a simplicial complex and M a differentiable manifold. A map f: $K \rightarrow M$ is called a *nice imbedding* if there is a simplicial complex L and an inclusion map, i: $K \rightarrow L$, a map g: $L \rightarrow M$ exhibiting L as a smooth triangulation of M such that

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & M \\ L \arrow[ur, "g"'] & \end{tikzcd}$$

is commutative.

Definition 2. Let f: $K \rightarrow M$ be a nice imbedding of a simplicial complex K in a differentiable manifold M. Then a submanifold $U \subseteq M$ is called a *simple neighborhood* of f if

(a) U is a compact manifold obtainable as the closure of an open set, int U, such that

$$f(K) \subset \operatorname{int} U;$$

(b) the map f: $K \rightarrow U$ obtained from f by considering U as range-space is a simple homotopy equivalence;

(c)

$$\pi_1(U - f(K), \partial U) = 0,$$
$$\pi_2(U - f(K), \partial U) = 0.$$

Condition (c) is needed to insure that the neighborhood is "highly connected at infinity." There are counter examples to the uniqueness theorem below if condition (c) is dropped from the definition of simple neighborhood.

Theorem (uniqueness of simple neighborhoods). *Let $n \geqq 6$. Let f: $K \rightarrow M^n$ be a nice imbedding of a simplicial complex into the interior of M^n, an n-dimensional differentiable manifold. Let U_1, U_2*

$\subseteq M^n$ be simple neighborhoods of f. Then there is an automorphism α: $M^n \rightarrow M^n$ satisfying these properties:

(a) *The automorphism α: $M^n \rightarrow M^n$ is isotopic to the identity* ($\alpha \approx 1$).

(b) *The diagram*

$$\begin{tikzcd} M^n \arrow[rr] & & M^n \\ & K \arrow[ul, "f"] \arrow[ur, "f"'] & \end{tikzcd}$$

is commutative.

(c) *The automorphism α gives rise to an isomorphism*

$$\alpha\colon U_1 \overset{\rightarrow}{\approx} U_2.$$

Some applications of the above uniqueness theorem are the following:

Corollary 1. *Let C^n be a compact contractible n-dimensional manifold such that ∂C^n is simply connected and $n \geqq 6$. Then C^n is isomorphic to an n-cell.*

Proof. If $p \in C^n$ is a point in the interior, then C^n itself is a simple neighborhood of p. Since there is a small n-cell C^n about p which is also a simple neighborhood of p, the uniqueness theorem applies, proving Corollary 1.

If one removes the hypothesis that ∂C^n be simply connected there are numerous differentiable manifolds distinct from D^n satisfying the remaining requirements.

Definition 3. A differentiable manifold W^n is called an s-cobordism between M_1^{n-1} and M_2^{n-1} if

$$\partial W^n = M_1 \cup M_2$$

the union being disjoint and the inclusion maps λ_i: $M_i \rightarrow W^n$ ($i = 1, 2$) are simply homotopy equivalences.

Corollary 2. *Let W^n be an s-cobordism between M_1 and M_2 where $n \geqq 6$. Then:*

$$W^n \approx M_1 \times I \approx M_2 \times I$$

and consequently:

$$M_1 \approx M_2.$$

PROOF. If W is the s-cobordism between M_1 and M_2, then let $[-1, +1]\times M_1$ be a collar neighborhood of M_1 in W such that $\{-1\}\times M_1$ is identified with M_1, then $[-1, +1]\times M_1$ and W both satisfy the conditions necessary to be simple neighborhoods of $\{0\}\times M_1$. Thus Corollary 2 follows.

An application of Corollary 2 and a theorem of Whitehead yields:

COROLLARY 3. *Let W^n be an h-cobordism between M_1 and M_2 where $n\geqq 6$ and W is simply connected. Then*

$$W \approx M_1 \times I \approx M_2 \times I$$

and consequently

$$M_1 \approx M_2.$$

It may be seen by example that the requirement of simple connectivity in Corollary 3 may not be dropped.

Actually, Corollary 2 (the s-cobordism theorem) may be strengthened somewhat using the minimum necessary hypotheses for the application of the uniqueness theorem:

COROLLARY 4. *Let W^n be a compact differentiable manifold, $n\geqq 6$. Let*

$$\partial W^n = M_1^{n-1} \cup M_2^{n-1}$$

the union being disjoint, such that:

(a) *the inclusion map*

$$\lambda_1\colon M_1 \to W^n$$

is a simple homotopy equivalence.

(b) *the relative sets*

$$\pi_i(W, M_2) = 0 \qquad \text{for } i = 1, 2.$$

Then:

$$W^n \approx M_1 \times I \approx M_2 \times I$$

and consequently:

$$M_1 \approx M_2.$$

Some lemmas useful for applications of simple neighborhoods are the following.

LEMMA 1. *Let $f\colon K\to W$ be a nice imbedding of K in W for which W is a simple neighborhood of K. Then*

$$\pi_q(W - K, \partial W) = 0 \qquad \textit{for all } q.$$

Proof. Let $U_K \subseteq W$ be a neighborhood of K where neighborhood is meant in the sense of [1]. In [1] it is proved that $U_K - K$ is homeomorphic with $\partial U_K \times [0, 1)$ and therefore the injection

$$(W - K, \partial W) \to (W - U_K, \partial W)$$

is a homotopy equivalence. We must show that $\pi_q(W - U_K, \partial W) = 0$ for all q. We have $\pi_q(W - U_K, \partial W) = 0$ for $q = 1, 2$ so by the relative Hurewicz theorem, it suffices to show that $H^q(W - U_K, \partial W) = 0$ for all q. By a standard duality theorem for manifolds,

$$H^q(W - U_K, \partial W) \approx H_{n-q}(W, U_K \cup \partial W).$$

Since the injection $U_K \to W$ is a homotopy equivalence, the group on the right is zero, proving Lemma 1.

Lemma 2. *Let $N \subseteq M \subseteq N'$ be differentiable manifolds and inclusion maps and let $K, L \subseteq N$ be subsets such that N, N' are simple neighborhoods of K and M is a simple neighborhood of L. Let the inclusion $i: L \to N$ be a simple homotopy equivalence. Then N' is a simple neighborhood of M which is a simple neighborhood of N which is a simple neighborhood of both K and L.*

Proof. All that needs to be checked is condition (c) since all other conditions of simple neighborhoods are implicit in the hypotheses of the lemma. I will prove condition (c) for the case of $M \subseteq N'$ all other cases being similar. It must be shown then that

$$\pi_q(N' - M, \partial N') = 0 \qquad \text{for } q = 1, 2.$$

Consider the exact relative homotopy sequence of the relative couple $\pi_{q+1}\{(N' - K, \partial N'), (N' - M, \partial N')\} \to \pi_{q+1}(N' - K, N' - M) \to \pi_q(N' - M, \partial N') \to \pi_q(N' - K, \partial N')$. Since N' is a simple neighborhood of K, both sets flanking $\pi_q(N' - M, \partial N')$ vanish, for all q (applying Lemma 1). This proves Lemma 2.

Proposition 1. *Let*

$$\phi: M_1^n \times D^2 \underset{\approx}{\rightarrow} M_2^n \times D^2$$

be a differentiable isomorphism with $n \geqq 5$. Then there is an isomorphism

$$\psi_0: M_1^n \underset{\approx}{\rightarrow} M_2^n.$$

(Moreover, there is a "bundle-map":

$$\begin{array}{ccc} M_1 \times D^2 & \xrightarrow{\psi} & M_2 \times D^2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\psi_0} & M_2 \end{array}$$

such that $\psi \approx \phi$.)

PROOF. Since ϕ: $M_1 \times D^2 \approx M_2 \times D^2$ is a differentiable isomorphism, it is a simple homotopy equivalence and it induces an isomorphism,

$$\partial\phi\colon M_1 \times S^1 \overset{\rightarrow}{\approx} M_2 \times S^1.$$

If $\pi_1(M_i \times S^1) \approx \pi_1(M_i) + \pi_1(S^1)$ for $i=1, 2$, then $\partial\phi$ induces an isomorphism

$$\partial\phi\colon \pi_1(M_1) \overset{\rightarrow}{\approx} \pi_1(M_2).$$

Let, then, $M_i \times R^1$ be the covering space of $M_i \times S^1$ associated to the subgroup $\pi_1(M_i) \subseteq \pi_1(M_i \times S^1)$ for $i=1, 2$. Then $\partial\phi$ gives rise to a differentiable isomorphism

$$\phi\colon M_1 \times R^1 \overset{\rightarrow}{\approx} M_2 \times R^1.$$

Identifying $M_1 \times R^1$ with $M_2 \times R^1$ via ϕ we may obtain positive numbers r_1, r_2, r_3, r_4 such that

$$M_1 \times D^1(r_1) \subseteq M_2 \times D^1(r_2) \subseteq M_1 \times D^1(r_3) \subseteq M_2 \times D^1(r_4).$$

Since ϕ: $M_1 \times D^2 \to M_2 \times D^2$ is a simple homotopy equivalence, the inclusion map, $M_1 \times D^1(r_1) \subseteq M_2 \times D^1(r_4)$ is also a simple homotopy equivalence, and the four above manifolds satisfy the conditions of Lemma 2. Then the conclusion of Lemma 2 gives a differentiable isomorphism

$$\phi^*\colon M_1 \times D^1(r_1) \overset{\rightarrow}{\approx} M_2 \times D^1(r_4)$$

implying the first assertion of the proposition applying the theorem of uniqueness of simple neighborhoods. The second assertion requires a slightly more detailed analysis.

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