MODEL THEORIES WITH TRUTH VALUES IN A UNIFORM SPACE

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In recent years the ultraproduct construction has been applied, e.g. in [4] and [2], to obtain a series of results in the theory of models for the ordinary two-valued first-order predicate logic. Most of the results in [4] and [2] have been generalized in [1] to predicate logic with truth values in the closed real unit interval. In this note we shall see that many of the methods and results of [4] and [2] and [1] can actually be extended to a very wide class of many-valued predicate logics, with truth values in any reasonably well-behaved compact Hausdorff uniform space.

We shall give a detailed statement of the definitions and two representative theorems. A complete account of the theory, including a number of generalizations of theorems from [2] and [1], as well as proofs, will appear in a future publication.

Let L be a formal system with the following symbols: a denumerable set V of individual variables, a set P of finitary predicates, a set C of finitary sentential connectives, a set Q of quantifier symbols, and distinguished symbols $e \in P$, $\& \in C$, $\exists \in Q$, where e and $\&$ are binary. Let the set F of formulas be the least set H such that

(i) $\{p(v_1, \cdots, v_n) \mid p \in P,\ p \text{ is } n\text{-ary},\ v_1, \cdots, v_n \in V\} \subseteq H$;

(ii) $\{c(\phi_1, \cdots, \phi_k) \mid c \in C,\ c \text{ is } k\text{-ary},\ \phi_1, \cdots, \phi_k \in H\} \subseteq H$;

(iii) $\{qv(\phi) \mid q \in Q,\ v \in V,\ \phi \in H\} \subseteq H$.

Free variables are defined as usual. ϕ is a sentence if $\phi \in F$ and ϕ has no free variables.

Given sets X, Y, and Z, $S(X)$ shall denote the set of all subsets of X and $f\colon Y \to Z$ shall mean f is a function on Y into Z.

If X is a uniform space with uniformity $\mathfrak{U}$ (see [3]), a set function $g\colon S(X) \to X$ is *uniformly continuous* if for each $U \in \mathfrak{U}$, there exists $U' \in \mathfrak{U}$ such that whenever $Y \subseteq X \cap U'[Z]$ and $Z \subseteq X \cap U'[Y]$, then $(g(Y), g(Z)) \in U$. $\mathfrak{X} = (X, f, t, \hat{c}, \hat{q})_{c \in C, q \in Q}$ is a *model theory* if

(i) X is a compact Hausdorff uniform space;

(ii) $f, t \in X$ and $f \neq t$;

(iii) for each k-ary $c \in C$, $\hat{c}\colon X^k \to X$ and $\hat{c}$ is continuous;

(iv) for each $q \in Q$, $\hat{q}\colon S(X) \to X$ and $\hat{q}$ is uniformly continuous.

$\mathfrak{A} = (A, p_{\mathfrak{A}})_{p \in P}$ is a *structure over* X if

(i) $A \neq 0$;

(ii) for each n-ary $p \in P$, $p_{\mathfrak{A}}\colon A^n \to X$;

(iii) for $a, b \in A$, $e_{\mathfrak{A}}(a, b) = t$ if $a = b$, and $e_{\mathfrak{A}}(a, b) = f$ if $a \neq b$.

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, in symbols $\mathfrak{A}\cong\mathfrak{B}$, if there is a 1-1 function h on A onto B such that for every n-ary $p\in P$ and all elements $a_1, \cdots, a_n\in A$, $p_{\mathfrak{A}}(a_1, \cdots, a_n)=p_{\mathfrak{B}}(ha_1, \cdots, ha_n)$.

For each $\phi\in F$ and $a\colon V\to A$, the value $\mathrm{Val}(\phi, \mathfrak{A}, a)\in X$ is defined inductively in the following manner:

(i) for each n-ary $p\in P$,

$$\mathrm{Val}(p(v_1, \cdots v_n), \mathfrak{A}, a) = p_{\mathfrak{A}}(a(v_1), \cdots, a(v_n));$$

(ii) for each k-ary $c\in C$, and each $\phi_1, \cdots, \phi_k\in F$,

$$\mathrm{Val}(c(\phi_1, \cdots, \phi_k), \mathfrak{A}, a) = \hat{c}(\mathrm{Val}(\phi_1, \mathfrak{A}, a), \cdots, \mathrm{Val}(\phi_k, \mathfrak{A}, a));$$

(iii) for each $q\in Q$, $\phi\in F$, and $v\in V$, $\mathrm{Val}(qv(\phi), \mathfrak{A}, a)=\hat{q}(Y)$ where

$$Y = \{\mathrm{Val}(\phi, \mathfrak{A}, b) \mid b\colon V \to A \text{ and } b(u) = a(u) \text{ whenever } u \neq v\}.$$

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent*, in symbols $\mathfrak{A}\equiv\mathfrak{B}$, if for every sentence ϕ, $\mathrm{Val}(\phi, \mathfrak{A})=\mathrm{Val}(\phi, \mathfrak{B})$.

Given structures $\mathfrak{A}_i=(A_i, p_i)_{p\in P}$, with $i\in I$, and an ultrafilter D on I, the set $A=\prod_{i\in I} A_i/D$ is defined as usual. Namely, for each function $f\in\prod_{i\in I} A_i$, we write

$$f/D = \left\{g \in \prod_{i\in I} A_i \,\middle|\, \{i \in I \mid f(i) = g(i)\} \in D\right\},$$

and we define

$$\prod_{i\in I} A_i/D = \left\{f/D \,\middle|\, f \in \prod_{i\in I} A_i\right\}.$$

The *ultraproduct* $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i/D$ of the structures $\mathfrak{A}_i$, with $i\in I$, is defined as follows: for each n-ary $p\in P$ and elements $h_1/D, \cdots, h_n/D$, $p_{\mathfrak{A}}(h_1/D, \cdots, h_n/D)$ is the unique $x\in X$ such that for each neighborhood N of x, $\{i \mid p_i(h_1(i), \cdots, h_n(i))\in N\}\in D$. We let $\mathfrak{A}^I/D$ denote the ultrapower of A. The following generalizes a theorem in [4] and [1].

COMPACTNESS THEOREM. *Let $\mathfrak{X}$ be a model theory. For each $i\in I$, let $x_i\in X$ and ϕ_i be a sentence. Let $J=\{j \mid j\subseteq I$ and j is finite$\}$. Suppose that whenever $i\in j\in J$,* $\mathrm{Val}(\phi_i, \mathfrak{A}_j)=x_i$. *Then there exists an ultrafilter D on J such that, for each $i\in I$,* $\mathrm{Val}(\phi_i, \prod_{j\in J}\mathfrak{A}_j/D)=x_i$.

A model theory $\mathfrak{X}$ is *good* if

(i) $\hat{\&}(x, y)=t$ if and only if $x=y=t$;

(ii) $t\in\hat{\exists}(Y)$ if and only if t is in the closure of Y;

(iii) if $x\neq y$ and $\phi\in F$, then there exists $\psi\in F$ such that

$$\{(\mathfrak{A}, a) \mid \mathrm{Val}(\phi, \mathfrak{A}, a) = x\} \subseteq \{(\mathfrak{A}, a) \mid \mathrm{Val}(\psi, \mathfrak{A}, a) = t\}$$

and

$$\{(\mathfrak{A}, a) \mid \mathrm{Val}(\phi, \mathfrak{A}, a) = y\} \cap \{(\mathfrak{A}, a) \mid \mathrm{Val}(\psi, \mathfrak{A}, a) = t\} = 0.$$

Examples of good model theories: (1) The two element Boolean algebra with the sup operator, $(\{0, 1\}, +, \cdot, ^-, 0, 1, \mathrm{Sup})$, with the discrete topology. (2) The MV-algebra on the closed real unit interval with the sup operator, $([0, 1], +, \cdot, ^-, 0, 1, \mathrm{Sup})$, with the usual topology.

We now generalize a theorem in [**2**; **1**].

Assume the generalized continuum hypothesis.

FUNDAMENTAL THEOREM. *Let $\mathfrak{X}$ be a good model theory, and $\mathfrak{A}$, $\mathfrak{B}$ be structures over $\mathfrak{X}$. Then $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A}^I/D \cong \mathfrak{B}^I/D$ for some set I and ultrafilter D on I.*

REFERENCES

1. C. C. Chang, *Theory of models of infinite valued logic.* I, II, III, Notices Amer. Math. Soc. **8** (1961), 68–69.

2. H. J. Keisler, *Ultraproducts and elementary classes*, Doctoral dissertation, Univ. of California, Berkeley, Calif., 1961. Also to appear in Indag. Math.

3. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

4. Anne C. Morel, D. Scott and A. Tarski, *Reduced products and the compactness theorem*, Notices Amer. Math. Soc. **5** (1958), 674–675.

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