SOME TWO-GENERATOR ONE-RELATOR
NON-HOPFIAN GROUPS

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In 1951 Graham Higman claimed (in [1]) that every finitely generated group with a single defining relation is Hopfian,\textsuperscript{2} attributing this fact to B. H. Neumann and Hanna Neumann. However we shall show that this is not, in any way, the case. For example the group

\begin{equation}
G = \langle a, b; a^{-1}b^3a = b^5 \rangle
\end{equation}

is non-Hopfian. Hence the following question of B. H. Neumann [2, p. 545] has a negative answer: Is every two-generator non-Hopfian group infinitely related?

This group $G$ turns out to be useful for deciding a somewhat different kind of question. For Graham Higman\textsuperscript{3} has pointed out that $G$ can, of course, be generated by $a$ and $b$. However it transpires that in terms of these generators $G$ requires \textit{more than one relation} to define it. Thus Higman has produced a counter-example to the following well-known conjecture: \textit{Let $G$ be generated by $n$ elements $a_1, a_2, \ldots, a_n$ and let $r$ be the least number in any set of defining relations between $a_1, a_2, \ldots, a_n$. Then $n - r$ is an invariant of $G$ (i.e. does not depend on the particular basis $a_1, a_2, \ldots, a_n$).} This conjecture has received some attention in the past; indeed there is a "proof" of it by Petresco [3].

The group defined by (1) is clearly only one of a larger family of groups of the kind

\begin{equation}
G = \langle a, b; a^{-1}b^m a = b^m \rangle.
\end{equation}

It is convenient at this point to introduce a definition. Thus we say two nonzero integers $l$ and $m$ are meshed if either

(i) $l$ or $m$ divides the other,

or,

(ii) $l$ and $m$ have precisely the same prime divisors. This definition enables us to distinguish easily between the Hopfian and the non-Hopfian groups in the family of groups (2). For the following theorem holds.

\textbf{Theorem 1.} \textit{Let $l$ and $m$ be nonzero integers. Then

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\textsuperscript{2} A group $G$ is Hopfian if $G/N \cong G$ implies $N = 1$; otherwise $G$ is non-Hopfian.
\textsuperscript{3} In a letter.}

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\[ G = \langle a, b; a^{-1}b^l a = b^m \rangle \]
is Hopfian if and only if \( l \) and \( m \) are meshed.

The proof of Theorem 1 is in three parts. Thus we prove

(a) if \( l \) or \( m \) divides the other, then \( G \) is residually finite\(^4\) and therefore Hopfian (Mal'cev [4]);
(b) if \( l \) and \( m \) are meshed but neither divides the other, then every epimorphism of \( G \) is an automorphism and so \( G \) is Hopfian;
(c) if \( l \) and \( m \) are not meshed, then \( G \) is non-Hopfian.

It is perhaps worthwhile to sketch the proof of (c). Here we may assume, without loss of generality, the existence of a prime \( p \) dividing \( l \) but not \( m \). Hence the mapping
\[ \eta: a \mapsto a, \quad b \mapsto b^p \]
defines an epimorphism of \( G \). Now it follows from the work of Magnus [5; 6] that
\[ [b^{l/p}, a]^{p^{l-m}} \neq 1. \]
However
\[ ([b^{l/p}, a]^{p^{l-m}})\eta = [b^l, a]^{p^{l-m}} = 1. \]
Therefore the kernel \( K \) of \( \eta \) is nontrivial and as
\[ G(\eta) \cong G/K \]
we have proved \( G \) is non-Hopfian.

The following theorem is a direct consequence of Theorem 1. It illustrates strikingly that hopficity is a finiteness condition of the weakest kind.

**Theorem 2.** The group
\[ G = \langle a, b; a^{-1}b^{12}a = b^{18} \rangle \]
is Hopfian but possesses a normal subgroup of finite index which is non-Hopfian.

It turns out that \( G'' \), the second derived group of
\[ G = \langle a, b; a^{-1}b^2 a = b^4 \rangle \]
is free. This fact enables us to prove the following theorem (cf. B H. Neumann [2, p. 544]).

\(^4\) \( G \) is residually finite if for each \( x \in G \) \((x \neq 1)\) there corresponds a normal subgroup \( N_x(G) \) such that \( G/N_x \) is finite and \( x \notin N_x \).
Theorem 3. The groups
\[ G = \langle a, b; a^{-1}b^2a = b^4 \rangle \]
and
\[ H = \langle c, d; c^{-1}d^2c = d^3, ([c, d]^2c^{-1})^2 = 1 \rangle \]
are homomorphic images of each other; however they are not isomorphic.

Finally we employ Theorem 1 to provide the first instance of a two-generator group which is soluble-of-length-three and non-Hopfian. Thus

Theorem 4. There exists a two-generator group which is soluble-of-length-three and non-Hopfian.

Theorem 4 may be compared with the results of B. H. Neumann and Hanna Neumann [7] and P. Hall [8].

References
4. A. I. Mal’cev, On isomorphic representations of infinite groups by matrices, Mat. Sb. 8 (1940), 405–422.