ON GROUPS WITH FINITELY MANY INDECOMPOSABLE INTEGRAL REPRESENTATIONS

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1. Introduction. The purpose of this note is to sketch a proof of the following theorem.

THEOREM. If $G$ is a finite group having finitely many non-isomorphic indecomposable integral representations then for no prime $p$ does $p^3$ divide the order of $G$.

It is known that the same hypothesis implies that all the Sylow subgroups of $G$ are cyclic; thus they are cyclic of order $p$ or $p^2$. We do not know whether the converse is true. On the other hand, we have shown elsewhere [1] that a cyclic group of order $p^2$ has finitely many non-isomorphic integral representations.

In the same place it is shown that the above theorem follows from this proposition:

PROPOSITION. Let $G$ be a cyclic group of order $p^3$. Then $G$ has infinitely many non-isomorphic indecomposable representations over the $p$-adic integers.

We outline below the proof of this proposition, which will appear in full elsewhere.

2. Construction of indecomposables. Let $\Lambda$ be a ring such that the Krull-Schmidt theorem holds for finitely generated left $\Lambda$-modules; this is certainly the case for algebras of finite rank over a complete valuation ring [3]. We shall write $\text{Hom}$ for $\text{Hom}_\Lambda$ and $\text{Ext}$ for $\text{Ext}_\Lambda$.

Suppose that $M$ and $N$ are indecomposable $\Lambda$-modules such that $\text{Hom}(M, N) = 0$, $\text{Hom}(N, M) = 0$. If $M^{(k)}$ is a direct sum of $k$ copies of $M$ then $\text{Hom}(M^{(k)}, M^{(k)})$ may be identified with the ring of $k \times k$ matrices with entries in $H = \text{Hom}(M, M)$. Also $\text{Ext}(N^{(u)}, M^{(v)})$ consists of $t \times u$ matrices with entries in $\text{Ext}(N, M)$. If $H' = \text{Hom}(N, N)$ then $\text{Ext}(N, M)$ is an $(H, H')$-bimodule, and $t \times t$ matrices over $H$ and $u \times u$ matrices over $H'$ operate in the obvious way on $\text{Ext}(N^{(u)}, M^{(v)})$.

We shall say that a matrix $X \in \text{Ext}(N^{(u)}, M^{(v)})$ is decomposable if there are invertible matrices $T$ over $H$ and $U$ over $H'$ such that

\[ X = TU. \]

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where, of course, $B$ and $D$ need not be square matrices.

**Lemma 1.** An extension $E$ of $N^{(u)}$ by $M^{(t)}$ with extension class $X$ is a decomposable module if and only if $X$ is a decomposable matrix.

In order to apply this lemma it is convenient to observe the following consequence.

**Corollary.** Let $\tilde{A}$, $\tilde{A}'$ be quotient rings of $H$, $H'$. Suppose $V \subseteq \text{Ext}(N, M)$ is an $(H, H')$-submodule and that $\tilde{V}$ is a quotient of $V$ on which $\tilde{A}$, $\tilde{A}'$ operate. If $X$ is a matrix with entries in $\tilde{V}$ whose image $\tilde{X}$ in $\tilde{V}$ is $(\tilde{A}, \tilde{A}^{'})$-indecomposable then the extension corresponding to $X$ is an indecomposable module.

3. **Construction of the submodule.** In this paragraph we set $\Delta = E_2 = Z_p^*G_p^*$, where $Z_p^*$ is the ring of $p$-adic integers, and $G_p^*$ is cyclic of order $p^2$ with generator $g$. We write $C = (g^p - 1)E_2$ and $E_1 = E_2/C$. For any module $N$, we shall set $\mathbf{N} = N/pN$.

Now $\text{Ext}(C, E_1) \approx \mathbb{Z}[g]/(g-1)^p$. We define $M$ to be the extension of $C$ by $E_1$ with extension class $g - 1$. Since $\text{Hom}(E_1, C) = 0$, $\text{Hom}(C, E_1) = 0$, we may apply Lemma 1 with $k = 1$. Thus $M$ is indecomposable. Further, if $H = \text{Hom}(M, M)$, there is a canonical monomorphism $\rho: H \rightarrow \text{Hom}(C, C) + \text{Hom}(E_1, E_1)$ whose image may be described as follows [2].

**Lemma 2.** $\rho(H)$ consists of pairs $(a_L, b_L)$, where $a, b \in E_2$ and the subscript $L$ denotes left multiplication, such that

$$(g - 1)(a - b) \in pE_2 + (g - 1)^pE_2.$$ 

Denoting by rad $H$ the Jacobson radical of $H$, we have the following consequence.

**Corollary.** $\rho(\text{rad} H)$ consists of pairs $(a_L, b_L) \subseteq \rho(H)$ such that $a, b \in \text{rad} E_2 = pE_2 + (g - 1)E_2$. Thus $\tilde{H} = H/\text{rad} H \approx \mathbb{Z}$.

Although $M$ is indecomposable this is not true of $\overline{M}$. We have instead the following result.

**Lemma 3.** $\overline{M} = E_2u \oplus E_2v$ as an $E_2$ module, where $pu = pv = (g - 1)u = (g - 1)p^{2-v}v = 0$.

Now let $V$ be the submodule $E_2u + E_2 (g - 1)v$ of $\overline{M}$. Then, as a consequence of Lemma 2, we have the following result.
Lemma 4. \( V \) is an \( H \)-submodule of \( \overline{M} \) and \( (\text{rad } H) V = E_2(g-1)^2v \). Thus \( \overline{V} = V/(\text{rad } H) V \) is a two-dimensional \( \overline{H} \)-space with basis \( \overline{u}, \overline{v} \), the images of \( u \) and \( (g-1)v \).

4. Proof of the proposition. We now change our notation so that \( A = E_3 = \mathbb{Z}^* G_p \) where \( G_p \) is cyclic of order \( p^z \) with generator \( g_p \). Then \( g_p \rightarrow g \) defines a ring epimorphism \( E_3 \rightarrow E_2 \); we use this to turn all \( E_3 \)-modules into \( E_2 \)-modules.

If \( N = (g_p^z - 1)E_3 \), and \( M \) is the module defined in §3, then \( \text{Hom}(M, N) = \text{Hom}(N, M) = 0 \) and \( \text{Ext}(N, M) = \overline{M} \). But \( H' = \text{Hom}(N, N) \) consists only of left multiplications \( aL, a \in E_3 \). Thus \( (\text{rad } H') V = E_2(g-1)^2v \) and \( \overline{H'} = H'/\text{rad } H' \approx \overline{Z} \) operates on \( \overline{V} \).

We are now in a position to apply the corollary to Lemma 1. For any integer \( k \) let \( X^{(k)} \in \text{Ext}(N^{(k)}, M^{(k)}) \) be the matrix \( X^{(k)} = uI + (g-1)vJ \), where \( J \) is any \( k \times k \) indecomposable matrix over \( \overline{Z} \). Since the matrices \( X^{(k)} = \overline{u}I + \overline{v}J \) are clearly \( \overline{Z} \)-indecomposable, i.e., \( (\overline{H}, \overline{H}') \)-indecomposable, the same must be true of the corresponding extensions.

References


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