A ZERO-ONE PROPERTY OF MIXING SEQUENCES OF EVENTS

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Communicated by J. L. Doob, February 27, 1962

A sequence of events \{A_n\} is mixing if for each event \(M, P(A_n M) - P(A_n) P(M) \to 0\); if it is also assumed that \(P(A_n) \to \alpha, 0 \leq \alpha \leq 1\), \{A_n\} is called mixing with density \(\alpha\). A sequence of events \{A_n\} is zero-one if its tail is trivial; semi-zero-one if every subsequence of \{A_n\} admits a zero-one subsequence.

THEOREM 1. A sequence of events is mixing if and only if it is semi-zero-one.

OUTLINE OF PROOF. Denote by \(\sigma_n, \sigma_\infty, \mathcal{C}_\infty\), respectively, the \(\sigma\)-fields generated by the event \(A_n\), by the events \(A_1, \ldots, A_n\), and by the events \(A_n, A_{n+1}, \ldots; \mathcal{C} = \bigcap_n \mathcal{C}_n\) is the tail of the sequence \{A_n\}. If \{A_n\} is zero-one, then for every bounded random variable \(X\) \(E(X/\mathcal{C}_n) \to E(X)\) with probability one and in \(L_1\) mean. One shows that if \(P(A_n) \to \alpha, 0 < \alpha < 1\), then \{A_n\} is mixing if and only if, for each bounded random variable \(X\), \(E(X/\mathcal{C}_n) \to E(X)\) in \(L_1\) mean (“in \(L_1\) mean” may here be replaced by “in probability” or by “uniformly except on a null event”). Hence a zero-one sequence, and also a semi-zero-one sequence, will be mixing. Now denote by \(A^*\) the event \(A\) or its complement and by \(I_A\) the characteristic function of \(A\). Let \{A_n\} be a sequence such that all events \(A_1^* \cdot A_n, n = 1, 2, \ldots\), are not null and let \(Q\) be the independent probability measure on \(\mathcal{C}_\infty\) with \(Q(A_n) = \alpha, n = 1, 2, \ldots, 0 < \alpha < 1\). Set

\[
X_n = \sum_{I_{A_1^* \cdots A_n^*}} \frac{P(A_1^* \cdots A_n^*)}{Q(A_1^* \cdots A_n^*)}, \quad n = 1, 2, \ldots,
\]

where the summation extends over all events \(A_1^* \cdot \cdots A_n^*\) of \(\mathcal{C}_\infty\). It is shown that every sequence of events mixing with density \(\alpha\) admits a subsequence \{A_n\} such that the \(X_n\)’s defined by (1) are uniformly integrable with respect to the measure \(Q\) (even uniformly bounded by \(1 - \epsilon, 1 + \epsilon\) where \(\epsilon\) is arbitrarily small). Doob’s discussion [1, pp. 343 ff.] shows that \(P\) is absolutely continuous with respect to \(Q\) on \(\mathcal{C}_\infty\); by Kolmogorov’s zero-one law \{A_n\} is \(Q\) zero-one, hence \{A_n\} is also \(P\) zero-one. It follows that a mixing sequence is semi-zero-one,

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1 This work was supported by the National Science Foundation Grant NSF G-14446.
unless it is mixing with density zero or one in which case one shows the existence of a zero-one subsequence by a direct argument.

From Theorem 1, one obtains that Kolmogorov automorphisms are mixing in all degrees [cf. Rokhlin 2, p. 14]. In probabilistic formulation: discrete-parameter stationary stochastic processes with trivial tail are mixing in all degrees. (The tail of a process \( \{X_n\}_{n=0}^{\infty} \) is the \( \sigma \)-field \( \bigcap_{n=0}^{\infty} \mathcal{C}_n \) where \( \mathcal{C}_n \) is generated by \( \cdots X_{n-1}, X_n \).) Via distribution functions Theorem 1 may also be applied to processes not necessarily stationary. A sequence of random variables \( \{X_n\}_{n=1}^{\infty} \) is called mixing if for some dense set \( D \) on the real line \( R \) the sequence of events \( \{A_n(y)\} \) is mixing for each \( y \in D \), where \( A_n(y) \) is defined on \( R \) by

\[
A_n(y) = [X_n < y], \quad n = 1, 2, \ldots.
\]

If \( \{A_n(y)\} \) is mixing with density \( F(y) \) for \( y \in D \), then \( F \) determines a distribution function \( F \) defined on \( R \) and \( P(A_n(y)) \) converges to \( F(y) \) on the continuity set of \( F(y) \); the sequence of random variables \( \{X_n\} \) is then called mixing with the limiting distribution function \( F(y) \); this last notion was introduced by Rényi [3]. It follows from Theorem 1 that if a sequence of random variables \( \{X_n\} \) is semi-zero-one, i.e. if every subsequence contains a subsequence with trivial tail, then \( \{X_n\} \) is mixing. It is further shown under rather weak assumptions that mixing is invariant under change of measure. A probability measure \( Q \) is semicontinuous with respect to \( P \) on a sequence of random variables \( \{X_n\} \) if every subsequence of \( \{X_n\} \) contains a further subsequence \( \{Y_n\} \) such that \( Q \) is absolutely continuous with respect to \( P \) on the tail of \( \{Y_n\} \).

**Theorem 2.** Let a sequence of random variables \( \{X_n\} \) be \( P \) mixing (with a limiting distribution function \( F(y) \)). If \( Q \) is a probability measure semicontinuous with respect to \( P \) on \( \{X_n\} \), then the sequence \( \{X_n\} \) is \( Q \) mixing (with the limiting distribution function \( F(y) \)).

In the proof, the invariance of mixing is obtained from Theorem 1 while the invariance of the limiting distribution is derived from the second theorem of Andersen and Jessen [4].

Theorem 2 extends Theorem 2 of Abbot and Blum [5] and certain results on invariance of limiting distributions of Rényi and Révész. Namely in Theorem 4 of Rényi [3] concerned with sums of independent random variables and in Examples 3 and 4 of Rényi and Révész [6] concerned with certain Markov chains, the premises may be weakened by assuming semicontinuity of \( Q \) with respect to \( P \) on the studied sequences of averages of random variables, instead
of absolute continuity of $Q$ with respect to $P$ on $\alpha$; the conclusions may be strengthened by asserting $Q$ mixing of these sequences with the limiting distribution function $F(y)$, instead of only the convergence of the distribution functions of the averages to $F(y)$.

**BIBLIOGRAPHY**


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THE EQUATION \((\partial^2/\partial x^2 + \partial^2/\partial y^2 + (x^2 + y^2)(\partial/\partial t))^2 u + \partial^2 u/\partial t^2 = f,\) WITH REAL COEFFICIENTS, IS "WITHOUT SOLUTIONS"

BY FRANÇOIS TREVES

Communicated by Lipman Bers, February 20, 1962

Indeed, the equation can be written \(PP^*(PP^*)^* u = f,\) where $P$ is Lewy's operator $\partial/\partial \bar{z} + i\bar{z}(\partial/\partial t),^2 \bar{z} = x + iy,$ and the star operation replaces the coefficients of a differential operator by their complex conjugates. Hörmander has shown\(^3\) that, whatever be the open set $\Omega$, there is a function $f \in C^\iota(\Omega)$ such that the equation $Pv = f$ does not have any distribution solution $v \in \mathcal{D}'(\Omega)$.

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