1. Symmetric invariants. Let $V$ be a finite-dimensional vector space over $\mathbb{R}$. Each $X \in V$ gives rise (by parallel translation) to a vector field on $V$ which we consider as a differential operator $\partial(X)$ on $V$. The mapping $X \mapsto \partial(X)$ extends to an isomorphism of the complex symmetric algebra $S(V)$ over $V$ onto the algebra of all differential operators on $V$ with constant complex coefficients. Let $G$ be a subgroup of the general linear group $\text{GL}(V)$. Let $I(V)$ denote the set of $G$-invariants in $S(V)$ and let $I_+(V)$ denote the set of $G$-invariants without constant term. The group $G$ acts on the dual space $V^*$ of $V$ by

$$ (g \cdot v^*)(v) = v^*(g^{-1}v), \quad g \in G, \quad v \in V, \quad v^* \in V^*, $$

and we can consider $S(V^*)$, $I(V^*)$, $I_+(V^*)$. An element $p \in S(V^*)$ (a polynomial function on $V$) is called $G$-harmonic if $\partial(J)p = 0$ for each $J \in I_+(V)$. Let $H(V^*)$ denote the set of $G$-harmonic polynomial functions.

Let $V^c$ denote the complexification of $V$. Suppose $B$ is a nondegenerate symmetric bilinear form on $V \times V$ and let $X \in V^c$ denote the linear form $Y \mapsto B(X, Y)$ on $V$. The mapping $X \mapsto X^*$ extends to an isomorphism $P \mapsto P^*$ of $S(V)$ onto $S(V^*)$. If $G$ leaves $B$ invariant then $I(V^*) = I_+(V^*)$.

We shall use the following notation: If $E$ and $F$ are linear subspaces of the associative algebra $A$ then $EF$ denotes the set of all sums $\sum_i e_i f_i$, ($e_i \in E$, $f_i \in F$).

**Theorem 1.** Let $B$ be a nondegenerate symmetric bilinear form on $V \times V$ and let $G$ be a Lie subgroup of $\text{GL}(V)$ leaving $B$ invariant. Suppose that either (1) $G$ is compact and $B$ positive definite or (2) $G$ is connected and semisimple. Then

$$ S(V^*) = I(V^*)H(V^*). $$

The case of a compact $G$ was noted independently by B. Kostant. It is a simple consequence of the fact that under the standard strictly positive definite inner product on $S(V^*)$ (invariant under $G$), the space $H(V^*)$ is the orthogonal complement to the ideal in $S(V^*)$ generated by $I_+(V^*)$. For the noncompact case, let $\mathfrak{g}$ denote the complexification of the Lie algebra of $G$. It is not difficult to prove that

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each compact real form \( u \) of \( g \) leaves invariant a real form \( W \) of \( V^c \) on which \( B \) is strictly positive definite. Now the compact case can be applied to the action of \( u \) on \( W \).

In the case when \( G \) is the orthogonal group \( O(n) \) acting on \( V = \mathbb{R}^n \) then \( I(V^*) \) consists of all polynomials in \( x_1^2 + \cdots + x_n^2 \) and \( H(V^*) \) consists of all the ordinary harmonic polynomials. Theorem 1 reduces to the classical fact that each \( p = p(x_1, \ldots, x_n) \) can be written \( p = \sum h_k \) where each \( h_k \) is harmonic. It is also known (compare Cartan [2, p. 285], Maass [9]) that \( H(V^*) \) is in this case spanned by the polynomials \( (a_1 x_1 + \cdots + a_n x_n)^k \) where \( a_1, \ldots, a_n \in \mathbb{C}, a_1^2 + \cdots + a_n^2 = 0 \) and \( k = 0, 1, \ldots \). The following generalization holds:

Theorem 2. Let the assumptions be as in Theorem 1. Let \( N_G \) denote the set of common zeros (in \( V^c \)) of the elements in \( I_+(V^*) \). Then \( H(V^*) \) is the direct sum

\[
H(V^*) = H_1(V^*) + H_2(V^*),
\]

where \( H_1(V^*) \) is the vector space spanned by the polynomials \( (X^*)^k \), \( (k = 0, 1, 2, \ldots, X \in N_G) \) and \( H_2(V^*) \) is the set of \( G \)-harmonic polynomials which vanish identically on \( N_G \).

For the case \( G = O(n) \) it follows easily from Hilbert's Nullstellensatz that \( H_2(V^*) = 0 \).

2. Exterior invariants. Let \( \Lambda(V) \) and \( \Lambda(V^*) \), respectively, denote the Grassmann algebras over the dual vector spaces \( V \) and \( V^* \). Each \( X \in V \) induces an antiderivation \( \delta(X) \) of \( \Lambda(V^*) \) given by

\[
\delta(X) \cdot (x_1 \wedge \cdots \wedge x_n) = \sum_{k=1}^{n} (-1)^{k+1} x_k(X) (x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_n)
\]

where \( \hat{x}_k \) indicates omission of \( x_k \). The mapping \( X \to \delta(X) \) extends uniquely to an isomorphism of \( \Lambda(V) \) into the algebra of all endomorphisms of \( \Lambda(V^*) \). Let \( G \) be any subgroup of \( GL(V) \). Let \( J(V) \) and \( J(V^*) \) denote the set of \( G \)-invariants in \( \Lambda(V) \) and \( \Lambda(V^*) \), respectively, \( J_+(V) \) and \( J_+(V^*) \) the sets of invariants without constant term. An element \( p \in \Lambda(V^*) \) is called \( G \)-primitive if \( \delta(J) p = 0 \) for each \( J \in J_+(V) \).

Let \( P(V^*) \) denote the set of \( G \)-primitive elements.

Theorem 3. Let the assumptions be as in Theorem 1. Then

\[
\Lambda(V^*) = J(V^*) \wedge P(V^*).
\]

Example. Let \( E \) be an \( n \)-dimensional Hilbert space over \( \mathbb{C} \). Considering \( E \) as a \( 2n \)-dimensional vector space \( V \) over \( \mathbb{R} \) the unitary
group $U(n)$ becomes a subgroup $G$ of the orthogonal group $O(2n)$. Let $Z_k = X_k + i Y_k$ ($1 \leq k \leq n$) be an orthonormal basis of $E$ and let $x_1, y_1, \ldots, x_n, y_n$ be the basis of $V^*$ dual to the basis $X_1, Y_1, \ldots, X_n, Y_n$ of $V$. It is easy to show that the element

$$ u = \sum_{i=1}^{n} x_k \wedge y_k $$

and its powers form a basis of $J_+(V^*)$. In view of Theorem 3 each $v \in \Lambda(V^*)$ can therefore be written

$$ v = \sum_{k} u^k \wedge p_k, $$

where each $p_k$ satisfies $\delta(u)p_k = 0$, (compare Weil [10, Théorème 3, p. 26]).

3. Invariants of Weyl groups. Let $u$ be an arbitrary semisimple Lie algebra over $R$ whose adjoint group $U$ is compact. Let $\theta$ be an arbitrary involutive automorphism of $u$ and let $u = \mathfrak{f} + \mathfrak{p}$ be the decomposition of $u$ into eigenspaces of $\theta$ for the eigenvalue $+1$ and $-1$ respectively. Let $K$ denote the analytic subgroup of $U$ corresponding to $\mathfrak{f}$. Let $h_0$ be a maximal abelian subspace of $\mathfrak{p}$ and extend $\mathfrak{h}_0$ to a maximal abelian subalgebra $\mathfrak{h}$ of $u$. The Weyl group of $\mathfrak{h}$ is defined as the group of linear transformations of $\mathfrak{h}$ induced by the set of elements in $U$ which leave $\mathfrak{h}$ invariant; the Weyl group of $\mathfrak{h}_0$ is defined as the group of linear transformations of $\mathfrak{h}_0$ induced by the set of elements in $K$ which leave $\mathfrak{h}_0$ invariant. Let these groups be denoted by $W(\mathfrak{h})$ and $W(\mathfrak{h}_0)$ and let $I(\mathfrak{h}^*)$ and $I(\mathfrak{h}_0^*)$ denote the corresponding sets of invariant polynomial functions. It is known that $W(\mathfrak{h}_0)$ can be described as the group of linear transformations of $\mathfrak{h}_0$ induced by those members of $W(\mathfrak{h})$ which leave $\mathfrak{h}_0$ invariant. Consequently, if the restriction to $\mathfrak{h}_0$ of a function $f$ on $\mathfrak{h}$ is denoted by $\tilde{f}$, the mapping $f \mapsto \tilde{f}$ maps $I(\mathfrak{h}^*)$ into $I(\mathfrak{h}_0^*)$.

Theorem 4. (i) Suppose $u$ is a classical compact simple Lie algebra and $\theta$ any involutive automorphism of $u$. Then the restriction mapping $f \mapsto \tilde{f}$ maps $I(\mathfrak{h}^*)$ onto $I(\mathfrak{h}_0^*)$.

(ii) Part (i) does not hold in general for the exceptional simple Lie algebras $u = \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$.

(iii) Let $Q(\mathfrak{h}^*)$ and $Q(\mathfrak{h}_0^*)$, respectively, denote the set of invariant rational functions on $\mathfrak{h}$ and $\mathfrak{h}_0$. Under the restriction mapping $f \mapsto \tilde{f}$, $Q(\mathfrak{h}^*)$ is mapped onto $Q(\mathfrak{h}_0^*)$.

Remarks. As $u$ and $\theta$ are arbitrary, $\mathfrak{f} + i\mathfrak{p}$ is the most general semisimple Lie algebra over $R$. Parts (i) and (ii) above therefore express
a property which is shared by all classical simple Lie algebras over \( \mathbb{R} \), yet fails to hold for all simple Lie algebras over \( \mathbb{R} \). Part (i) is proved by verification using Cartan’s classification \([1]\) of the root structures of \( U \) and of the symmetric space \( U/K \). Since the groups \( W(\mathfrak{h}) \) and \( W(\mathfrak{h}) \) are finite groups generated by reflections, \( I(\mathfrak{h}^*) \) and \( I(\mathfrak{h}^*) \) are polynomial rings, (Chevalley \([4]\)). The degrees of the generators can be readily determined from known facts. It is then found that if the space \( U/K \) is \( E_8/F_4 \), \( E_7/(E_6 \times T) \) or \( E_6/(E_7 \times SU(2)) \), the ring \( I(\mathfrak{h}^*) \) contains a homogeneous element of degree 3, 4, and 6, respectively, which cannot be obtained from \( I(\mathfrak{h}^*) \) by restriction. Part (iii) had been proved independently by Harish-Chandra.

4. Fundamental functions on quadrics. Let \( G \) be a topological group, \( H \) a closed subgroup, \( G/H \) the set of left cosets \( gH \) with the natural topology. If \( f \) is a complex-valued continuous function on \( G/H \) and \( x \in G \) then \( f^x \) denotes the function on \( G/H \) given by \( f^x(gH) = f(xgH) \ (g \in G) \). The function \( f \) is called fundamental (Cartan \([3, p. 218]\)) if the vector space \( V_f \) over \( \mathbb{C} \) spanned by the functions \( f^x \ (x \in G) \) is finite-dimensional.

Consider the quadric \( C_{p,q} \subseteq \mathbb{R}^{p+q} \) given by the equation

\[
Q(X) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1, \quad (p \geq 1, q \geq 0).
\]

Let \( O(p, q) \) denote the group of linear transformations of \( \mathbb{R}^{p+q} \) leaving \( Q \) invariant. The group \( O(p, q) \) acts transitively on \( C_{p,q} \) and the subgroup leaving \((1, 0, \ldots, 0)\) fixed is isomorphic to \( O(p - 1, q) \) so we make the identification

\[
(1) \quad C_{p,q} = O(p, q)/O(p - 1, q).
\]

It is obvious that if \( P = P(x_1, \ldots, x_{p+q}) \) is a polynomial then the restriction of \( P \) to \( C_{p,q} \) is a fundamental function. On the other hand we have

**Theorem 5.** Let \( f \) be a fundamental function on \( C_{p,q} \). Assume \( (p, q) \neq (1, 1) \). Then there exists a polynomial \( P = P(x_1, \ldots, x_{p+q}) \) such that

\[
f = P \quad \text{on } C_{p,q}.
\]

**Remarks.** 1. The special case \( q = 0 \) (for which \( O(p, q) \) is compact) was already proved by Hecke \([6]\) and Cartan \([3]\).

2. If \( p = 1 \), the denominator in (1) is compact and by use of a compact real form of the complexification of the Lie algebra of \( O(1, q) \) this case can be reduced to the case 1. This procedure fails
for \( q = 1 \) because \( O(1, 1) \) is not semisimple and the theorem fails to hold for \( (p, q) = (1, 1) \) as the example \( f(x_1, x_2) = \cosh^{-1}(|x_1|) \) shows. The case \( (p, q) = (1, 2) \) was settled by Loewner [8] using special features of the Lobatchefsky plane.

3. By a method of descent the remaining cases can be reduced to the case \( x_1^2 + x_2^2 - x_3^2 = 1 \) (which differs radically from the case \( x_1^2 - x_2^2 - x_3^2 = 1 \) by the noncompactness of the isotropy group). Here one can make use of the special property of the identity component of the group \( O(2, 1) \), namely that every representation of it extends to a representation of the corresponding complex subgroup of \( GL(3, \mathbb{C}) \), (see Harish-Chandra [5]).

4. From Theorem 1 it is clear that the polynomial \( P \) can be taken to be an \( O(p, q) \)-harmonic polynomial, that is a polynomial satisfying the equation

\[
\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right) P = 0.
\]

It follows that the function \( f \) is necessarily a sum of eigenfunctions of the Laplace-Beltrami operator on \( C_{p,q} \) (formed by means of the indefinite Riemannian metric on \( C_{p,q} \) [7]).

BIBLIOGRAPHY


Massachusetts Institute of Technology