ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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1. Classification of certain spaces of continuously differentiable
functions of two variables. Denote by $C_0$ the space of all complex-
valued continuous functions on the plane that are zero at infinity.
Write $\| \cdot \|_\infty$ for the supremum norm on $C_0$. Denote by $D$ the dense
subspace of $C_0$ consisting of infinitely differentiable functions with
compact support.

Throughout we shall be concerned with differential operators of the
form

$$\sum a_{m,n} \frac{\partial^{m+n}}{\partial x^m \partial y^n};$$

the $a_{m,n}$ are complex constants. For each set $\mathcal{A}$ of such operators, we
define $C_0(\mathcal{A})$ to be the space of all $f$ in $C_0$ having $Af$ in $C_0$ (in the sense
of Laurent Schwartz) for all $A$ in $\mathcal{A}$. Equivalently, $C_0(\mathcal{A})$ is the com-
pletion of $D$ under the seminorms

$$f \mapsto \| f \|_\infty \quad \text{and} \quad f \mapsto \| Af \|_\infty, \quad A \text{ in } \mathcal{A}.$$

Each $C_0(\mathcal{A})$ so defined is a translation-invariant space of functions;
those that are furthermore invariant under rotations of the plane
will be called rotating spaces of differentiable functions.

Certain of these spaces are familiar, namely the spaces $C^N_0$ consist-
ing of those functions in $C_0$ that have all derivatives of order $\leq N$ in
$C_0$, and the space $C^\infty_0$, which is $\cap_N C^N_0$. A rotating space of differenti-
able functions will be called proper if it is not one of the $C^N_0$ and not
$C^\infty_0$. Here is the classification of such spaces.

We use the notation

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

**Theorem 1.1.** If $\mathcal{A}$ is a proper subset of

$$\{ \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} : m + n = N \},$$

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for $N$ a positive integer, then $C_0(\alpha)$ is a proper rotating space of differentiable functions between $C_0^n$ and $C_0^{n-1}$. If $\alpha_1$ and $\alpha_2$ are distinct subsets of (1.2) then $C_0(\alpha_1)$ and $C_0(\alpha_2)$ are distinct. Each proper rotating space of differentiable functions is a $C_0(\alpha)$, where $\alpha$ is a proper subset of (1.2) for some $N$.

**Corollary 1.2.** Every proper rotating space of differentiable functions is a Banach algebra under pointwise multiplication.

For instance, the six distinct proper rotating spaces of differentiable functions between $C_0$ and $C_0^2$ are

\[
\begin{align*}
C_0(\partial^2/\partial z^2), \\
C_0(\partial^2/\partial \bar{z}^2), \\
C_0(\partial^2/\partial z\partial \bar{z}) = C_0(\partial^2/\partial x^2 + \partial^2/\partial y^2), \\
C_0(\partial^2/\partial x^2, \partial^2/\partial y^2) = C_0(\partial^2/\partial x^2 - \partial^2/\partial y^2, \partial^2/\partial x\partial y), \\
C_0(\partial^2/\partial z^2, \partial^2/\partial z\partial \bar{z}), \\
C_0(\partial^2/\partial \bar{z}^2, \partial^2/\partial z\partial \bar{z}).
\end{align*}
\]

**Remark 1.3.** Suppose instead of the supremum norm we take as basic norm $\|f\|_p = \left\{ \int |f|^p \right\}^{1/p}$, $1 < p < \infty$. Then there are no proper rotating spaces

$$L_p(\alpha) = \{ f : f \in L_p, Af \in L_p \text{ for all } A \in \alpha \}.$$  

Indeed the only rotating spaces are the Sobolev spaces

$$L^N_p = \{ f : f \in L_p, (\partial^{m+n}/\partial x^m \partial y^n)f \in L_p, m + n \leq N \},$$

analogous of the $C_0^N$ and $L_p^\infty$, which is identical with $C_0^\infty$.

In a sense, rotating spaces are the ones that have geometrical significance. From this point of view the correct definition in norm $\| \cdot \|_\infty$ of "Sobolev space" would include not only the $C_0^\infty$ but also the proper rotating spaces.

2. **Spaces of continuously differentiable functions on Riemann surfaces.** It is possible to define algebras of functions on Riemann surfaces corresponding to those described in Theorem 1.1 and to show that these algebras determine the conformal structure.

Let $U$ be an open subset of the plane, $C(U)$ the space of all complex-valued continuous functions on $U$. For $\alpha$ a set of differential operators of the form (1.1), we denote by $C(U, \alpha)$ the subspace of $C(U)$ consisting of those $f$ in $C(U)$ with $Af$ in $C(U)$ (in the sense of Laurent Schwartz) for each $A$ in $\alpha$.

For a general set $\alpha$ of differential operators, the spaces $C(U, \alpha)$
have no interesting invariance properties. However we have

**Lemma 2.1.** If $\alpha$ is a subset of (1.2), the spaces $C(U, \alpha)$ are invariant under conformal transformations.

This result allows the extension of the definition of the $C(U, \alpha)$ to Riemann surfaces. If $R$ is a Riemann surface and $\alpha$ a subset of (1.2), $C(R, \alpha)$ is defined to be the space of those functions on $R$ such that if $U = \{z: |z| < 1\}$ and $\phi: U \rightarrow R$ is a coordinate disk, the composite function $f \circ \phi$ is in $C(U, \alpha)$.

Each $C(R, \alpha)$ is an algebra of functions on $R$, with multiplicative linear functionals corresponding to points of $R$, and with a natural complete locally convex topology. In this topology $C(R, \alpha)$ is a Banach algebra if and only if $R$ is compact.

In three instances $C(R, \alpha)$ can be described in terms of exterior differential operators defined globally on $R$.

$$C(R, \partial/\partial z) = \{f: f \text{ and } df \text{ continuous on } R\},$$

$$C(R, \partial/\partial \bar{z}) = \{f: f \text{ and } \bar{df} \text{ continuous on } R\},$$

$$C(R, \partial^2/\partial z \partial \bar{z}) = \{f: f \text{ and } \Delta f \text{ continuous on } R\},$$

where the operators $\partial$, $\bar{\partial}$ and $\Delta$, taking functions into differential forms, are defined in terms of any coordinate system by

$$\partial f = \frac{\partial f}{\partial z} dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}, \quad \Delta f = \frac{1}{4} \frac{\partial^2 f}{\partial z \partial \bar{z}} dz d\bar{z}.$$

The following result states the extent to which the algebras $C(R, \alpha)$ determine the conformal structure of $R$.

**Theorem 2.2.** Let $R_1$ and $R_2$ be connected Riemann surfaces and $\alpha$ be a proper subset of (1.2). Each conformal equivalence of $R_1$ with $R_2$ induces an algebra isomorphism of $C(R_1, \alpha)$ with $C(R_2, \alpha)$. If $\alpha$ is symmetric (i.e., $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ in $\alpha$ if $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ in $\alpha$), an anticonformal equivalence of $R_1$ with $R_2$ also induces an algebra isomorphism of $C(R_1, \alpha)$ with $C(R_2, \alpha)$. No other algebra isomorphisms of $C(R_1, \alpha)$ with $C(R_2, \alpha)$ are possible.

A similar result identifies all algebra homomorphisms of the $C(R, \alpha)$.

3. **Sup norm estimates.** The work of the preceding sections is based on the existence and nonexistence of certain sup norm estimates for constant-coefficient differential operators. In this section we state
these results, which may be of some independent interest. All these results remain valid for \( n \) variables.

If \( P \) is a polynomial,

\[
P(x, y) = \sum a_{m,n}x^m y^n,
\]
we denote by \( P^N \) its homogeneous part of degree \( N \),

\[
P^N(x, y) = \sum_{m+n=N} a_{m,n}x^m y^n,
\]
and by \( \hat{P} \) its Fourier transform, the differential operator

\[
\sum (-i)^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n}.
\]

An operator \( \hat{P} \) of order \( N \) is called elliptic if \( P^N(x, y) \neq 0 \) for \( (x, y) \neq (0, 0) \).

**Theorem 3.1.** Let \( Q, P_1, \ldots, P_r \) be polynomials of degree \( N \) or less. Then the following are equivalent:

1. There is a constant \( K \) so that

\[
\|\hat{Q}f\|_{\infty} \leq K(\|\hat{P}_1f\|_{\infty} + \cdots + \|\hat{P}_r f\|_{\infty})
\]

for all \( f \) in \( D \).

2. There are finite measures \( \mu_1, \ldots, \mu_r \) in the plane whose Fourier-Stieltjes transforms \( \hat{\mu}_1, \ldots, \hat{\mu}_r \) satisfy

\[
Q = P_1\hat{\mu}_1 + \cdots + P_r\hat{\mu}_r.
\]

If (1) and (2) hold, it is possible to find constants \( c_1, \ldots, c_r \) so that

\[
Q^N = c_1P_1^N + \cdots + c_rP_r^N.
\]

**Theorem 3.2.** Let \( \hat{P} \) be elliptic and of order \( N \geq 2 \). Then for each \( \hat{Q} \) of order strictly less than \( N \) there is a constant \( K \) so that

\[
\|\hat{Q}f\|_{\infty} \leq K(\|\hat{P}f\|_{\infty} + \|f\|_{\infty})
\]

for all \( f \) in \( D \).

**Corollary 3.3.** \( \partial^{m+n}/\partial x^m \partial \bar{z}^n \) is elliptic. Hence, if \( p+q < m+n \), there is a constant \( K \) so that

\[
\|(\partial^{m+n}/\partial x^p \partial y^q)f\|_{\infty} \leq K(\|(\partial^{m+n}/\partial x^p \partial \bar{z}^q)f\|_{\infty} + \|f\|_{\infty})
\]

for all \( f \) in \( D \).

**Remark 3.4.** The situation for estimates in \( \| \cdot \|_p \), \( 1 < p < \infty \), is quite different, which is the reason for the phenomenon mentioned in Re-
mark 1.3. To be precise, if $\hat{P}$ is elliptic of order $N$, $\hat{Q}$ arbitrary of order $\leq N$, it is known that there is an estimate of the form

$$\|\hat{Q}f\|_p \leq K(\|\hat{P}f\|_p + \|f\|_p)$$

and the existence of these estimates is characteristic of ellipticity. It is natural to ask whether the existence of sup norm estimates like those in Theorem 3.2 (where $\hat{Q}$ has strictly lower order) also characterizes ellipticity. In the plane the answer is no, but in higher dimensions yes.

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