TWO APPLICATIONS OF THE METHOD OF CONSTRUCTION
BY ULTRAPOWERS TO ANALYSIS

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1. Introduction. Recently, A. Robinson in [1] has given a proper extension of classical analysis, which he called nonstandard analysis. His theory is based on the general metamathematical result that there exist nonstandard models for the system $\mathbb{R}$ of real numbers. Such models of $\mathbb{R}$ may be constructed in the form of ultrapowers as defined by T. Frayne, D. Scott, and A. Tarski in [2]. The object of this paper is to apply Robinson's method in order to obtain a new proof of the Hahn-Banach extension theorem and in order to give a new and simple proof of a result about the existence of certain measures on Boolean algebras which was recently obtained by O. Nikodem in [3; 4].

It may be of interest to the reader to point out that the use of nonstandard arguments in the proof of the Hahn-Banach extension theorem eliminates the use of Zorn's lemma. In fact, the validity of the Hahn-Banach extension theorem is a consequence of the apparently weaker hypothesis that every proper filter is contained in an ultrafilter, i.e., the prime ideal theorem for Boolean algebras. It seems likely, that conversely the Hahn-Banach extension theorem implies the prime ideal theorem for Boolean algebras.

A more detailed presentation of the subject of this announcement will be contained in lecture notes on nonstandard analysis under preparation by the author.

2. Nonstandard models of $\mathbb{R}$. Let $\mathbb{R}$ denote the real number system. Let $D$ be an arbitrary set and let $\mathcal{U}$ be an ultrafilter on $D$. If $A$ and $B$ are two mappings of $D$ into $\mathbb{R}$, i.e., $A, B \in D^\mathbb{R}$, then we say that $A \equiv_\mathcal{U} B$ if and only if $\{ n: n \in D \text{ and } A(n) = B(n) \} \subseteq \mathcal{U}$. The relation $A \equiv_\mathcal{U} B$ is easily seen to be an equivalence relation. The set $D^\mathbb{R}/\mathcal{U}$ of all equivalence classes will be denoted by $\mathbb{R}^*$ and the equivalence class of a mapping $A$ of $D$ into $\mathbb{R}$ will be denoted by $a$. Thus $A \in a$. Finally, we define the algebraic operations in $\mathbb{R}^*$ as follows: $a + b = c$ if and only if there exist elements $A \in a$, $B \in b$ and $C \in c$ such that $\{ n: n \in D \text{ and } A(n) + B(n) = C(n) \} \subseteq \mathcal{U}$; and a similar definition

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for \( ab = c \) and \( a \leq b \). With these definitions \( R^* \) is a totally-ordered field and \( R \subseteq R^* \). If \( R \not= R^* \), then \( R^* \) is non-archimedean and is a non-standard model of \( R \). In this case the following two subsets are introduced. \( M_0 \) is the set of all \( a \in R^* \) such that \( |a| < r \) for some \( r \in R \). Then \( M_0 \) is a ring and the elements of \( M_0 \) are called the finite elements of \( R^* \). \( M_1 \) is the set of all \( a \in R^* \) such that \( |a| < r \) for all \( r \in R \) and \( r > 0 \). The elements of \( M_1 \) are called infinitesimals. Furthermore, \( M_1 \) is a maximal ideal of \( M_0 \) and \( M_0/M_1 \) is isomorphic to \( R \). The homomorphism of \( M_0 \) onto \( R \) with kernel \( M_1 \) will be called “standard part” and will be denoted by \( \text{st} \). If \( a \in M_0 \), then \( \text{st}(a) \) is the unique real number which is infinitely close to \( a \). This homomorphism is order preserving.

The terminology used in this section is taken from [1].

3. The Hahn-Banach extension theorem. In this section we shall sketch a proof of the Hahn-Banach extension theorem using non-standard arguments.

Theorem (Hahn-Banach). Let \( E \) be a real linear space and let \( p \) be a sublinear functional defined on \( E \), i.e., a mapping \( p \) of \( E \) into \( R \) such that \( p(x+y) \leq p(x) + p(y) \) for all \( x, y \in E \) and \( p(tx) = tp(x) \) for all \( x \in E \) and all real \( t \geq 0 \). If \( f \) is a real linear functional defined on a linear subspace \( G \) of \( E \) such that \( f(x) \leq p(x) \) for all \( x \in G \), then there exists a real linear functional \( F \) on \( E \) such that \( F(x) = f(x) \) for all \( x \in G \) and \( F(x) \leq p(x) \) for all \( x \in E \).

Proof. Let \( \{f_n : n \in D\} \) be the family of all linear functionals which are defined on some linear subspace of \( E \) which contains \( G \) and which have the following properties: \( f_n(x) = f(x) \) for all \( x \in G \) and \( f_n(x) \leq p(x) \) for all \( x \in E \) for which \( f_n(x) \) is defined. It is evident that \( D \neq \emptyset \). For every \( x \in E \) we denote by \( D_x \) the set of all indices \( n \in D \) such that the domain of \( f_n \) contains \( x \). It follows from Banach’s proof (see [5, p. 28]) that \( D_x \neq \emptyset \) for all \( x \in E \). Furthermore, the family \( \{D_x : x \in E\} \) of subsets of \( D \) has the finite intersection property, i.e., if \( x_1, \ldots, x_n \) are elements of \( E \), then \( \bigcap_{i=1}^{n} D_{x_i} \neq \emptyset \). Indeed, apply Banach’s construction successively to the elements \( x_1, \ldots, x_n \). Hence, there exists an ultrafilter \( U \) on \( D \) which contains the family \( \{D_x : x \in E\} \). Let \( R^* \) be the ultrapower \( D^R/U \). Then we define the following mapping \( \bar{f} \) of \( E \) into \( R^* \). If \( x \in E \), then \( \bar{f}(x) \) is that element of \( R^* \) which is determined by an element \( A \) of \( D^R \) such that \( A(n) = f_n(x) \) for all \( n \in D_x \). Then it is easy to see that \( \bar{f} \) is a linear transformation of \( E \) into \( R^* \) (consider \( R^* \) as a vector space over \( R \)) and that \( \bar{f} \) has the following properties: (i) \( \bar{f}(x) = f(x) \) for all \( x \in G \) and (ii) \( \bar{f}(x) \leq p(x) \) for all \( x \in E \).
From (ii) it follows that $-p(-x) \leq \hat{f}(x) \leq p(x)$ for all $x \in E$, i.e., $\hat{f}(x)$ is finite for all $x \in E$. Hence, $F(x) = \text{st}(\hat{f}(x))$ is the required linear functional. This completes the proof of the theorem.

Remark. The proof shows that the ultrafilter $U$ is fixed, i.e., there exists an element $n \in D$ such that $\{n\} \in U$. Furthermore, there exists a one-to-one correspondence between the family of all ultrafilters on $D$ containing the family $\{D_x : x \in E\}$ and the family of all extensions of $f$ satisfying the conditions of the theorem.

4. A theorem of Nikodým. Let $B$ be a Boolean algebra. It is well-known that there does not always exist on $B$ a strictly positive real-valued finitely additive measure. Therefore, the following result, which was recently obtained by O. Nikodým in [3; 4], is of interest.

Theorem (O. Nikodym). For every Boolean algebra $B$ there exists a totally ordered field $F$ which is in general non-archimedean such that $B$ admits a strictly positive $F$-valued finitely additive measure.

Proof. Let $B$ be a Boolean algebra and let $\{\mu_n : n \in D\}$ be the collection of all real-valued measures on $B$ such that $\mu_n(1) = 1$ for all $n \in D$. For every $0 \not= a \in B$ we denote by $D_a$ the set of all $n \in D$ such that $\mu_n(a) \neq 0$. It is well known that $D \neq \emptyset$ for all $0 \not= a \in B$ (Stone's Theorem). Hence, the family of sets $\{D_a : 0 \not= a \in B\}$ has the finite intersection property. Let $U$ be an ultrafilter on $D$ which contains the family $\{D_a : 0 \not= a \in B\}$. Let $F$ be the ultrapower $D^R/U$. Then $F$ is a totally ordered field, $R \subseteq F$ and $R \not= F$ if and only if $F$ is non-archimedean. We define now the following mapping $\bar{\mu}$ of $B$ into $F$. If $0 \not= a \in B$, then $\bar{\mu}(a)$ is that element of $F$ which is determined by the element $A \in D^R$ which has the following property: $A(n) = \mu_n(a)$ for all $n \in D_a$; and we define $\bar{\mu}(0) = 0$. Then, by construction, $\bar{\mu}$ has the following properties: (i) $\bar{\mu}(a) = 0$ if and only if $a = 0$, i.e., $\bar{\mu}$ is strictly positive and (ii) $\bar{\mu}(a \lor b) = \bar{\mu}(a) + \bar{\mu}(b)$ whenever $a \land b = 0$, i.e., $\bar{\mu}$ is finitely-additive. This completes the proof of the theorem.

Remark. If $B$ does not admit a strictly positive real-valued finitely additive measure, then the totally-ordered field $F$ constructed in the proof of the preceding theorem is a proper extension of $R$ and hence, $\bar{\mu}(a)$ is infinitesimal for at least one element $0 \not= a \in B$.

References

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