RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

TCHEBYCHEFF APPROXIMATION IN LOCALLY CONVEX SPACES

BY E. W. CHENEY AND A. A. GOLDSTEIN

Communicated by Creighton Buck, April 24, 1962

The characterization—via extremal properties—of the polynomials of best uniform approximation has been extensively treated in the literature, e.g. [1–5]. Offered here (Theorem 1) is an abstract form of the Tchebycheff-Bernstein Theorem, free of finite-dimensionality and unisolvence restrictions.

THEOREM 1. *Let X denote a nonvoid compact subset of a locally convex linear topological space E, and let Φ denote a continuous real-valued function on X. In order that an element F of E^* minimize the expression*

$$\Delta(F) = \max_{x\in X} \mid F(x) - \Phi(x) \mid$$

it is necessary and sufficient that 0 *belong to the convex closure of the set* $\{x\in X\colon F(x)-\Phi(x)=\Delta(F)\}\cup-\{x\in X\colon \Phi(x)-F(x)=\Delta(F)\}$.

This theorem may be proved directly or reduced to a special case of the following theorem, which is itself an abstract treatment of the "linear programming" problem. Both theorems are proved by means of the standard separation theorem for convex sets [6, p. 418].

THEOREM 2. *Let X be a nonvoid compact subset in a locally convex linear topological space E, with* 0 *not in the convex closure of X. Let Φ be a lower semi-continuous real-valued function on X and v a prescribed element of E. In order that an element F of E^* maximize the expression $F(v)$ under the restriction*

$$\max_{x\in X}\{F(x) - \Phi(x)\} \leqq 0$$

it is necessary and sufficient that v be a non-negative multiple of a point in the convex closure of the set $\{x\in X\colon F(x)=\Phi(x)\}$.

It will now be indicated how the Tchebycheff-Bernstein Theorem may be derived from Theorem 1. A set of continuous functions

$\{f_1, \cdots, f_n\}$ defined on a closed interval $[a, b]$ is termed a *Tchebycheff system* if for every choice of n distinct points $x_j \in [a, b]$, $\det f_i(x_j) \neq 0$.

LEMMA. *Let $\{f_1, \cdots, f_n\}$ be a Tchebycheff system on $[a, b]$. For each $x \in [a, b]$ define an n-tuple $\hat{x} = (f_1(x), \cdots, f_n(x))$. Let $a \leqq x_0 < x_1 < \cdots < x_n \leqq b$ and let $\lambda_0, \cdots, \lambda_n$ be nonzero constants. In order that* 0 *belong to the convex hull of $\{\lambda_0 \hat{x}_0, \cdots, \lambda_n \hat{x}_n\}$ it is necessary and sufficient that $\lambda_i \lambda_{i-1} < 0$ for $i = 1, \cdots, n$.*

THEOREM 3 [1, p. 3]. *Let $\{f_1, \cdots, f_n\}$ be a Tchebycheff system on $[a, b]$. In order that a generalized polynomial $\sum c_i f_i$ be a best uniform approximation to a prescribed element g of $C[a, b]$ it is necessary and sufficient that the function $e = g - \sum c_i f_i$ exhibit $n+1$ oscillations thus: $e(x_i) = -e(x_{i-1}) = \pm \|e\|$, $a \leqq x_0 < \cdots < x_n \leqq b$.*

PROOF. The map $x \rightarrow \hat{x}$ defined in the lemma is clearly continuous. Thus the set $X = \{\hat{x}: a \leqq x \leqq b\}$ is compact in E_n. The map is 1-1, for otherwise the postulate for a Tchebycheff family is violated. Hence the map is a homeomorphism, and consequently the equation $\Phi(\hat{x}) = g(x)$ defines Φ as a continuous real-valued function on X. Identifying the function $\sum c_i f_i$ with the linear functional $F = (c_1, \cdots, c_n)$ on E_n, we have $\Delta(F) = \max_{x \in X} |F(x) - \Phi(x)| = \max_{a \leqq x \leqq b} |\sum c_i f_i(x) - g(x)|$. By Theorem 1, $\sum c_i f_i$ will be a best approximation to g iff 0 belongs to the convex closure of the set $Y = \{\hat{x}: F(\hat{x}) - \Phi(\hat{x}) = \Delta(F)\} \cup -\{\hat{x}: \Phi(\hat{x}) - F(\hat{x}) = \Delta(F)\}$. Since Y is compact so is its convex hull. By a theorem of Carathéodory, the origin is in the convex closure of Y iff there exist $r \leqq n+1$ points of Y whose convex hull contains 0. From the fact that $\{f_1, \cdots, f_n\}$ is a Tchebycheff system, $r = n+1$. An appeal to the lemma completes the proof.

REFERENCES

1. S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Gauthier-Villars, Paris, 1926.

2. E. Ya Remez, *Methods of best approximate representation of functions in the sense of Tchebycheff*, Kiev, 1935. (Ukrainian)

3. L. G. Schnirelmann, *Sur les approximations uniformes*, Bull. Acad. Sci. URSS Ser. Math. Nr. 1 (1938), 53–59.

4. A. N. Kolmogorov, *Note on the subject of P. L. Tchebycheff's polynomials which deviate least from a given function*, Uspehi Mat. Nauk **3** (1948), no. 1(23), 216–222.

5. S. T. Zuhovickii, *Approximation of real functions in the Tchebycheff sense*, Uspehi Mat. Nauk **11** (1956), no. 2(68), 125–159.

6. N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1958.

IOWA STATE UNIVERSITY AND
MASSACHUSETTS INSTITUTE OF TECHNOLOGY