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SOME CONVOLUTION ALGEBRAS OF MEASURES ON [1, ∞) AND A REPRESENTATION THEOREM FOR LAPLACE-STIELTJES TRANSFORMS

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1. Introduction. In [1] we studied the set $\mathcal{A}_R$ of power series $\sum_{n=1}^{\infty} a_n z^n$ convergent for $|z| < R$, $0 < R \leq 1$, under the multiplication

$$
\left( \sum_{n=1}^{\infty} a_n z^n \right) \left( \sum_{n=1}^{\infty} b_n z^n \right) = \sum_{n=1}^{\infty} \left( \sum_{r+s=n} a_r b_s \right) z^n.
$$

It was found that $\mathcal{A}_R$, with the usual addition and scalar multiplication, and with the topology of uniform convergence on compact subsets of the disk $|z| < R$, is a locally convex algebra with identity. Also $\sum_{n=1}^{\infty} a_n z^n$ is invertible (has an inverse in $\mathcal{A}_R$ with respect to the above multiplication) if and only if $a_1 \neq 0$. As a consequence we obtained the following expansion theorem for analytic functions (E. Hille [2]).

**Theorem.** Let $f(z)$ be analytic for $|z| < R$, $0 < R \leq 1$, with $f(0) = 0$. Then associated with any function $g(z)$ analytic in $|z| < R$ with the properties $g(0) = 0$, $g'(0) \neq 0$, there is a unique expansion of the form

$$
f(z) = \sum_{n=1}^{\infty} c_n g(z^n), \quad |z| < R.
$$

Our object in this paper is to obtain an analogous result for Laplace-Stieltjes integrals (Theorem 1 below). We shall base the discussion on the theory of convolution algebras of complex measures on $[0, \infty)$.
as described in [3]. More precisely, we shall need the adaptation of this theory to the multiplicative semi-group \([1, \infty)\).

2. Convolution algebras depending on a weight function. In this section we record as Proposition 1 the appropriate modifications of the needed portions of [3].

**Proposition 1.** Let \(\phi(t)\) be a real-valued Borel measurable function defined on \([1, \infty)\) satisfying

\[
0 < \phi(t_1 t_2) \leq \phi(t_1) \phi(t_2), \quad t_1, t_2 \geq 1; \quad \phi(1) = 1.
\]

Let \(\mathcal{B}\) be the ring of bounded Borel subsets of \([1, \infty)\), and let \(S(\phi)\) denote the set of complex measures \(\mu\) on \(\mathcal{B}\) such that

\[
\int_1^\infty \phi(t) d\mu(t) < \infty.
\]

Finally let

\[
\omega = \inf \left\{ \log \phi(t) / \log t \mid t > 1 \right\}.
\]

If \(\omega = -\infty\), then \(\mu\) is invertible if and only if \(\mu(\{1\}) \neq 0\), \(\mu \in S(\phi)\).

**Proof.** Let \(\mathcal{B}'\) be the ring of bounded Borel subsets of \([0, \infty)\). Let the function \(\phi'\) be defined by \(\phi'(t) = \phi(e^t), \ t \geq 0\). Then \(\phi'\) satisfies the requirements given in [3] for a weight function for the additive semi-group \([0, \infty)\). Hence the set \(S'(\phi')\) of complex measures on \(\mathcal{B}'\) with finite \(\phi'\)-norms is a commutative Banach algebra with identity defined by a unit mass at 0. The proof of this statement, given in [3], can readily be adapted to \(S(\phi)\). Let us, however, note that \(S'(\phi')\) and \(S(\phi)\) are isomorphic. Indeed, the exponential function provides an isomorphism of the underlying semi-groups with preservation of bounded Borel sets \(B' \leftrightarrow B\). This induces the one to one correspondence of measures

\[
a' \leftrightarrow a, \quad a'(B') = a(B); \quad a' \in S'(\phi'), \quad a \in S(\phi), \quad B' \in \mathcal{B}', \quad B \in \mathcal{B},
\]

and this correspondence preserves addition, scalar multiplication, convolution, total variation, and norm. Therefore \(S(\phi)\) is a Banach
algebra as described. To prove the last assertion we note that 
\( \omega = \inf \{ \log \phi'(t)/t \mid t > 0 \} \). The assertion now follows from Theorem 4.18.5 of \([3]\) and the isomorphism (5).

3. **The special case** \( \phi_\sigma(t) = e^{-\sigma(t-1)}, \sigma > 0, \) **and Laplace-Stieltjes transforms.** We have for \( t_1, t_2 \geq 1 \)

\[
\phi_\sigma(t_1 t_2) = e^{-\sigma(t_1 t_2 - 1)} \leq e^{-\sigma(t_1 t_2 - 2)} = \phi_\sigma(t_1) \phi_\sigma(t_2),
\]

and it is clear that \( \phi_\sigma \) is a suitable weight function for \([1, \infty)\). Furthermore

\[
\log \phi_\sigma(t)/\log t = -\sigma(t - 1)/\log t \to -\infty \quad \text{as} \quad t \to \infty.
\]

Therefore we have the following result.

**Proposition 2.** \( S(\phi_\sigma) \) is a Banach algebra of the type described in Proposition 1. The invertible elements \( a \) of \( S(\phi_\sigma) \) are characterized by the condition \( a(\{1\}) \neq 0 \).

We can now establish the representation theorem alluded to in the Introduction.

**Theorem 1.** Let \( \sigma > 0 \). Let

\[
f(s) = \int_1^\infty e^{-st}da(t), \quad g(s) = \int_1^\infty e^{-st}db(t), \quad \Re s \geq \sigma,
\]

where \( a \) and \( b \) are complex measures on \( B \), and the integrals are absolutely convergent. Then

\[
e^sg(s) \to b(\{1\}) \quad \text{as} \quad \Re s \to \infty.
\]

If this limit is not zero, there is a complex measure \( c \) on \( B \) such that

\[
f(s) = \int_1^\infty g(st)dc(t), \quad \Re s \geq \sigma,
\]

the integral converging absolutely.

**Proof.** For \( \Re s \geq \sigma \) we have \( e^sg(s) = \int_1^\infty e^{-st}db(t) \). But \( e^{-\sigma(t-1)} \to \chi_{\{1\}}(t) \) as \( \Re s \to \infty \). Thus we obtain (7) by Lebesgue's dominated convergence theorem. By (6), \( a, b \in S(\phi_\sigma) \), and by Proposition 2 and our assumption regarding (7), \( b \) is invertible. Hence there is a unique \( c \in S(\phi_\sigma) \) such that \( a = bc \). From this equation and the basic definition (3) we conclude

\[
\int_1^\infty e^{-st}da(t) = \int_1^\infty \int_1^\infty e^{-sxy}d[b \times c](x, y), \quad \Re s \geq \sigma.
\]
By the Fubini theorem this equation implies (8).

4. Remarks. If the integrals in (6) are absolutely convergent in an open half-plane \( \Re s > \rho, \rho \geq 0 \), then for every \( \sigma > \rho \) there is a measure \( c_\sigma \) such that (8) holds. To show that \( c_\sigma \) is independent of \( \sigma \) we reason as follows. \( a \) and \( b \) belong to the set of measures \( S_\rho = \bigcap_{\sigma > \rho} S(\phi_\sigma) \). In each algebra \( S(\phi_\sigma), \sigma > \rho \), \( b \) has an inverse, but since these algebras are linearly ordered by inclusion, all these inverses are the same. Thus \( b^{-1} \) exists in \( S_\rho \). But then \( c = b^{-1}a \) also belongs to \( S_\rho \). Hence we have a single formula (8) holding in the given half-plane \( \Re s > \rho \). The set \( S_\rho \) is the analog of the set \( a_R \) of power series. Like \( a_R, S_\rho \) is a complete, countably-normed algebra.

In the representation (8) of \( f(s) \), we have regarded \( g \) as having been given, and \( c \) as having been determined. We can reverse the situation and choose any \( c \) subject to the condition \( c(\{1\}) \neq 0 \). The equation \( a = bc \) is then uniquely solvable for \( b \) in the appropriate algebra. (8) now follows as before, \( g \) being the transform of \( b \).

References


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