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SOME CONVOLUTION ALGEBRAS OF MEASURES ON $[1, \infty)$ AND A REPRESENTATION THEOREM FOR LAPLACE-STIELTJES TRANSFORMS

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1. **Introduction.** In [1] we studied the set $\mathfrak{A}_R$ of power series $\sum_{n=1}^{\infty} a_n z^n$ convergent for $|z| < R$, $0 < R \leqq 1$, under the multiplication

$$\left(\sum_{n=1}^{\infty} a_n z^n\right)\left(\sum_{n=1}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty}\left(\sum_{rs=n} a_r b_s\right) z^n.$$

It was found that $\mathfrak{A}_R$, with the usual addition and scalar multiplication, and with the topology of uniform convergence on compact subsets of the disk $|z| < R$, is a locally convex algebra with identity. Also $\sum_{n=1}^{\infty} a_n z^n$ is invertible (has an inverse in $\mathfrak{A}_R$ with respect to the above multiplication) if and only if $a_1 \neq 0$. As a consequence we obtained the following expansion theorem for analytic functions (E. Hille [2]).

THEOREM. *Let $f(z)$ be analytic for $|z| < R$, $0 < R \leqq 1$, with $f(0) = 0$. Then associated with any function $g(z)$ analytic in $|z| < R$ with the properties $g(0) = 0$, $g'(0) \neq 0$, there is a unique expansion of the form*

$$f(z) = \sum_{n=1}^{\infty} c_n g(z^n), \qquad |z| < R.$$

Our object in this paper is to obtain an analogous result for Laplace-Stieltjes integrals (Theorem 1 below). We shall base the discussion on the theory of convolution algebras of complex measures on $[0, \infty)$

as described in [3]. More precisely, we shall need the adaptation of this theory to the multiplicative semi-group $[1, \infty)$.

2. **Convolution algebras depending on a weight function.** In this section we record as Proposition 1 the appropriate modifications of the needed portions of [3].

PROPOSITION 1. *Let $\phi(t)$ be a real-valued Borel measurable function defined on $[1, \infty)$ satisfying*

$$(1) \qquad 0 < \phi(t_1t_2) \leqq \phi(t_1)\phi(t_2),\ t_1,\ t_2 \geqq 1; \qquad \phi(1) = 1.$$

Let $\mathcal{B}$ be the ring of bounded Borel subsets of $[1, \infty)$, and let $S(\phi)$ denote the set of complex measures a on $\mathcal{B}$ such that

$$(2) \qquad \|a\| = \int_1^\infty \phi(t)d\,|\,a\,|\,(t) < \infty.$$

Finally let

$$(3) \quad [ab](B) = [a \times b](\{(x, y)\,|\,xy \in B,\ x \geqq 1,\ y \geqq 1\}); \qquad a,\ b \in S(\phi),\ B \in \mathcal{B}.$$

Then $S(\phi)$ is a commutative Banach algebra with norm defined by (2), *with multiplication defined by* (3), *with the obvious definitions of addition and scalar multiplication, and with identity defined by a unit mass at* 1. *Let*

$$(4) \qquad \omega = \inf\{\log \phi(t)/\log t\,|\,t > 1\}.$$

If $\omega = -\infty$, then a is invertible if and only if $a(\{1\}) \neq 0$, $a \in S(\phi)$.

PROOF. Let $\mathcal{B}'$ be the ring of bounded Borel subsets of $[0, \infty)$. Let the function ϕ' be defined by $\phi'(t) = \phi(e^t)$, $t \geqq 0$. Then ϕ' satisfies the requirements given in [3] for a weight function for the additive semi-group $[0, \infty)$. Hence the set $S'(\phi')$ of complex measures on $\mathcal{B}'$ with finite ϕ'-norms is a commutative Banach algebra with identity defined by a unit mass at 0. The proof of this statement, given in [3], can readily be adapted to $S(\phi)$. Let us, however, note that $S'(\phi')$ and $S(\phi)$ are isomorphic. Indeed, the exponential function provides an isomorphism of the underlying semi-groups with preservation of bounded Borel sets $B' \leftrightarrow B$. This induces the one to one correspondence of measures

$$(5) \quad a' \leftrightarrow a,\ a'(B') = a(B);\ a' \in S'(\phi'),\ a \in S(\phi),\ B' \in \mathcal{B}',\ B \in \mathcal{B},$$

and this correspondence preserves addition, scalar multiplication, convolution, total variation, and norm. Therefore $S(\phi)$ is a Banach algebra as described. To prove the last assertion we note that $\omega=\inf\{\log \phi'(t)/t \mid t>0\}$. The assertion now follows from Theorem 4.18.5 of [3] and the isomorphism (5).

3. **The special case $\phi_\sigma(t)=e^{-\sigma(t-1)}$, $\sigma>0$, and Laplace-Stieltjes transforms.** We have for $t_1, t_2 \geqq 1$

$$\phi_\sigma(t_1t_2) = e^{-\sigma(t_1t_2-1)} \leqq e^{-\sigma(t_1+t_2-2)} = \phi_\sigma(t_1)\phi_\sigma(t_2),$$

and it is clear that ϕ_σ is a suitable weight function for $[1, \infty)$. Furthermore

$$\log \phi_\sigma(t)/\log t = -\sigma(t-1)/\log t \to -\infty \text{ as } t\to\infty.$$

Therefore we have the following result.

PROPOSITION 2. *$S(\phi_\sigma)$ is a Banach algebra of the type described in Proposition 1. The invertible elements a of $S(\phi_\sigma)$ are characterized by the condition $a(\{1\})\neq 0$.*

We can now establish the representation theorem alluded to in the Introduction.

THEOREM 1. *Let $\sigma>0$. Let*

$$f(s) = \int_1^\infty e^{-st}da(t), \quad g(s) = \int_1^\infty e^{-st}db(t), \quad \operatorname{Re} s \geqq \sigma, \tag{6}$$

where a and b are complex measures on $\mathcal{B}$, and the integrals are absolutely convergent. Then

$$e^s g(s) \to b(\{1\}) \text{ as } \operatorname{Re} s \to \infty. \tag{7}$$

If this limit is not zero, there is a complex measure c on $\mathcal{B}$ such that

$$f(s) = \int_1^\infty g(st)dc(t), \qquad \operatorname{Re} s \geqq \sigma, \tag{8}$$

the integral converging absolutely.

PROOF. For $\operatorname{Re} s\geqq\sigma$ we have $e^s g(s)=\int_1^\infty e^{-s(t-1)}db(t)$. But $e^{-s(t-1)} \to \chi_{\{1\}}(t)$ as $\operatorname{Re} s\to\infty$. Thus we obtain (7) by Lebesgue's dominated convergence theorem. By (6), $a, b\in S(\phi_\sigma)$, and by Proposition 2 and our assumption regarding (7), b is invertible. Hence there is a unique $c\in S(\phi_\sigma)$ such that $a=bc$. From this equation and the basic definition (3) we conclude

$$\int_1^\infty e^{-st}da(t) = \int_1^\infty \int_1^\infty e^{-sxy}d[b\times c](x, y), \qquad \operatorname{Re} s \geqq \sigma.$$

By the Fubini theorem this equation implies (8).

4. **Remarks.** If the integrals in (6) are absolutely convergent in an open half-plane Re $s > \rho$, $\rho \geqq 0$, then for every $\sigma > \rho$ there is a measure c_σ such that (8) holds. To show that c_σ is independent of σ we reason as follows. a and b belong to the set of measures $S_\rho = \bigcap_{\sigma > \rho} S(\phi_\sigma)$. In each algebra $S(\phi_\sigma)$, $\sigma > \rho$, b has an inverse, but since these algebras are linearly ordered by inclusion, all these inverses are the same. Thus b^{-1} exists in S_ρ. But then $c = b^{-1}a$ also belongs to S_ρ. Hence we have a single formula (8) holding in the given half-plane Re $s > \rho$. The set S_ρ is the analog of the set $\mathfrak{A}_R$ of power series. Like $\mathfrak{A}_R$, S_ρ is a complete, countably-normed algebra.

In the representation (8) of $f(s)$, we have regarded g as having been given, and c as having been determined. We can reverse the situation and choose any c subject to the condition $c(\{1\}) \neq 0$. The equation $a = bc$ is then uniquely solvable for b in the appropriate algebra. (8) now follows as before, g being the transform of b.

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