DIFFERENTIABLE OPEN MAPS[1]

BY P. T. CHURCH

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Let $f: M^n \to N^n$ be a continuous function, where M^n and N^n are n-manifolds (without boundary). It will be implicitly assumed that the manifolds share the differentiability properties of f, e.g., $f \in C'$ implies that M^n and N^n are C' manifolds. The map f is called *open* if, whenever U is open in M^n, $f(U)$ is open in N^n; it is *light* if, for every $y \in N^n$, $\dim(f^{-1}(y)) \leqq 0$.

For $n=2$, it is well-known that a nonconstant complex analytic function is open and light. Conversely, Stoilow proved that every light open map is locally, at each point, topologically equivalent [**9**, p. 198] to one of the canonical analytic maps g_d, defined by $g_d(z) = z^d$ ($d=1, 2, \cdots$). If it is not assumed that f is light, however, f may be quite different from a g_d. R. D. Anderson in [**1**] (see also [**2**]) constructed an open map $f: S^2 \to S^2$ such that, for each $y \in S^2$, $f^{-1}(y)$ is a nondegenerate continuum.

For $n \geqq 2$, let $F_{n,d}: E^n \to E^n$ be the canonical open map defined by: $F_{n,d}(x_1, x_2, \cdots, x_n) = (u_1, u_2, \cdots, u_n)$, where $u_1 + iu_2 = (x_1 + ix_2)^d$ ($i = \sqrt{-1}$) and $u_j = x_j$ ($j = 3, 4, \cdots, n$; $d = 1, 2, \cdots$). Since each $F_{n,d}$ is a generalization of g_d, it is natural to wonder (for $n \geqq 3$) how much an arbitrary open map f, satisfying some additional condition, differs locally from one of them.

The *branch set* B_f is the set of points in M^n at which f fails to be a local homeomorphism (defined in [**3**]).

THEOREM. *Let $f: M^n \to N^n$ be C^n and open ($n \geqq 2$), where M^n is compact or f is light. Then there exists a closed set E, $\dim E \leqq n-3$, such that, for each x in $M^n - E$, there exists a neighborhood of x on which f is topologically equivalent to one of the canonical maps $F_{n,d}$ ($d=1, 2, \cdots$). Moreover, E is nowhere dense in B_f unless f is a local homeomorphism.*

In particular, for $n=2$ we have the classical structure. In [**4**, p. 620, (4.3)] there is a 2-to-1 open map $f: S^5 \to S^5$ for which B_f is not locally a manifold at any point (it is necessarily [**4**, p. 620, (4.2)] a 3-gm mod 2); thus *some* differentiability assumption is required above. There is a C^∞ open map $f: E^2 \to E^2$ for which B_f is the y-axis; thus either compactness of the domain or lightness of the map is needed. An example $f: E^3 \to E^3$ (or $f: S^3 \to S^3$) given by E. Hemmingsen

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and the author in [4, p. 620, (3.3)] indicates the extent of possible pathology. There B_f has a Cantor set of point components, so that the exceptional set E in the Theorem is necessary (f can be shown to be topologically equivalent to a C^∞ map).

The following corollary is a generalization of the inverse function theorem. Let Z be the set of zeros of the Jacobian determinant.

COROLLARY. *If* $f: E^n \to E^n$, $n \geqq 3$, $f \in C^n$, *and* $\dim Z \leqq 0$, *then* f *is a local homeomorphism.*

PROOF. The map f is light, and its Jacobian determinant is either non-negative or nonpositive everywhere. Thus f is open [8], and the result follows from the Theorem. More generally, the conclusion holds if $\dim (B_f) \leqq 0$.

A basic lemma for the proof of the Theorem follows. The set of points in M^n at which the Jacobian matrix has rank at most q is denoted by R_q.

LEMMA. *Let* $h: M^n \to N^p$, *where* $h \in C^n$ *and* M^n *and* N^p *are* n- *and* p-*manifolds, respectively. Then* $\dim (f(R_q)) \leqq q$.

In particular, $\dim(h(M^n)) \leqq n$. The lemma is related to the theorem of A. P. Morse [6] on the image of the critical set of a real-valued function, and to Sard's Theorem [7]. If f is light, then [5, pp. 91–92] $\dim (R_q) \leqq q$.

The proof of the Theorem employs Morse's Theorem, a uniform form of the implicit function theorem, and some results from [3]. Detailed proofs will appear elsewhere.

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SYRACUSE UNIVERSITY