SIMULTANEOUS APPROXIMATION AND ALGEBRAIC INDEPENDENCE OF NUMBERS

BY WOLFGANG M. SCHMIDT

Communicated by Paul T. Bateman, April 14, 1962

1. **The result.** Write $\|\xi\|$ for the distance of a real number ξ to the nearest integer. Given a polynomial f in any number of variables, put $L(f)$ for the sum of the absolute values of its coefficients. Our main result is the following

THEOREM. *Let $f(x_1, \cdots, x_n)$ be a nonzero polynomial with integral coefficients and of degree d_k in x_k $(k=1, \cdots, n)$. Let $Q>0$ and $q\neq 0$ be integers and $\xi_1, \cdots, \xi_n$ reals satisfying*

$$(1) \qquad |q| \leqq Q, \qquad |\xi_k q| \leqq Q \qquad (k = 1, \cdots, n)$$

and

$$(2) \quad \|\xi_k q\| \leqq \|\xi_1 q\|^{d_1} \cdots \|\xi_{k-1} q\|^{d_{k-1}} Q^{1-d_1-\cdots-d_n}[d_k 2^{d_k+\cdots+d_n+k} L(f)]^{-1} \qquad (k = 1, 2, \cdots, n).^1$$

Then

$$(3) \qquad |f(\xi_1, \cdots, \xi_n)| \geqq \|\xi_1 q\|^{d_1} \cdots \|\xi_n q\|^{d_n} q^{-d_1-\cdots-d_n} 2^{-n}.$$

Generalizing a well-known result of Liouville [1] for $n=1$ (for further references, see [4]), we obtain a

SUFFICIENT CONDITION FOR ALGEBRAIC INDEPENDENCE. *Let $\xi_1, \cdots, \xi_n$ be an n-tuple of reals such that to every $d>0$ there is an integer q with*

$$(4) \qquad 0 < \|\xi_k q\| < (\|\xi_1 q\| \cdots \|\xi_{k-1} q\|)^d q^{1-nd} \qquad (k = 1, 2, \cdots, n).^1$$

Then $\xi_1, \cdots, \xi_n$ are algebraically independent (over the rationals).

This is true since (2) can be satisfied with

$$Q \geqq \max(q, |\xi_1 q|, \cdots, |\xi_n q|)$$

and for every f, if (4) can be satisfied for every d.

For example, the numbers $\xi_1, \xi_2, \cdots$ defined by

$$\xi_k = \sum_{t=1}^{\infty} 2^{-(kt)!} \qquad (k = 1, 2, \cdots)$$

are independent.[2] It suffices to show that $\xi_1, \cdots, \xi_n$ are independent

[1] For $k=1$, the product of the $\|\xi_i q\|$'s means 1.

[2] A similar system of algebraically independent numbers was given by von Neumann [3]. Actually von Neumann's system has the power of the continuum.

Added in proof. Using our criterion and the sequence $q_t = 2^{(t^t)}$ one can show that the numbers $\xi(x) = \sum_{t=1}^{\infty} 2^{-[t^{t+x}]}$, $0 \leqq x < 1$, are algebraically independent. This example (with $0 < x \leqq 1$) is due to H. Kneser (Bull. Soc. Math. Belg. 12 (1960), 23–27).

for fixed n. Take a positive integer h with $n!|h$ and put $q_h=2^{h!}$. Then

$$\|\xi_k q_h\| = 2^{h!}\sum_{t=h/k+1}^{\infty} 2^{-(kt)!} \qquad (k=1,\cdots,n),$$

which yields

$$2^{h!-(h+k)!} < \|\xi_k q_h\| < 2^{1+h!-(h+k)!} \qquad (k=1,\cdots,n)$$

and

$$(\|\xi_1 q_k\| \cdots \|\xi_{k-1}q_k\|)^d q_h^{-nd} > 2^{-nd(h+k-1)!-ndh!} \qquad (k=1,\cdots,n).$$

Hence q_h for $h>h(d)$ satisfies our condition, and the assertion is proved.

2. **The proof.** Let $Q>0$, $q\neq 0$, $p_1,\cdots,p_n$ be integers such that

$$|q| \leqq Q, \qquad |p_1| \leqq Q,\cdots, |p_n| \leqq Q.$$

Let f be a polynomial as described in the theorem, and put $L_Q(f) = L(f(Qx_1,\cdots,Qx_n))$.

We define now recursively a set of $2n+1$ polynomials

$$f_n, g_n, \cdots, f_1, g_1, f_0$$

as follows:

$$f_n(x_1,\cdots,x_n) = q^{d_1+\cdots+d_n} f(x_1/q,\cdots,x_n/q);$$

if $f_k(x_1,\cdots,x_k)$ is already defined, let δ_k be the largest non-negative integer such that $(x_k-p_k)^{\delta_k}|f_k(x_1,\cdots,x_k)$, and then put

$$g_k(x_1,\cdots,x_k) = (x_k-p_k)^{-\delta_k} f_k(x_1,\cdots,x_k)$$

and

$$f_{k-1}(x_1,\cdots,x_{k-1}) = g_k(x_1,\cdots,x_{k-1},p_k).$$

It is clear that these polynomials are $\neq 0$.

From the definition,

$$L_Q(f_n) = \sum_{i_1}\cdots\sum_{i_n} |a_{i_1\cdots i_n}| q^{d_1+\cdots+d_n-i_1-\cdots-i_n} Q^{i_1+\cdots+i_n}$$

and hence

$$L_Q(f_n) \leqq Q^{d_1+\cdots+d_n} L(f). \tag{5}$$

Assume next that, say,

$$f_k(x_1,\cdots,x_k) = \sum_{i_1=0}^{d_1}\cdots\sum_{i_k=0}^{d_k} a(k)_{i_1\cdots i_k} x_1^{i_1}\cdots x_k^{i_k}$$

and

$$g_k(x_1, \cdots, x_k) = \sum_{i_1=0}^{d_1} \cdots \sum_{i_k=0}^{d_k} b(k)_{i_1 \cdots i_k} x_1^{i_1} \cdots x_k^{i_k}.$$

It is then clear from the definition of g_k in terms of f_k that, for every choice of $i_1, \cdots, i_{k-1}$,

$$\sum_{i_k=0}^{d_k} a(k)_{i_1 \cdots i_k} x_k^{i_k} \equiv (x_k - p_k)^{\delta_k} \sum_{i_k=0}^{d_k} b(k)_{i_1 \cdots i_k} x_k^{i_k}.$$

An inequality of Mahler [2] states that

$$\prod_{\sigma=1}^{s} L(g_\sigma) \leqq 2^d L\left(\prod_{\sigma=1}^{s} g_\sigma \right),$$

where $g_1(x), \cdots, g_s(x)$ are polynomials in one variable and d is the degree of their product. Putting $s=2$, $g_1(x) = (Qx - p_k)^{\delta_k}$, $g_2(x) = \sum_{i_k=0}^{d_k} b(k)_{i_1 \cdots i_k} Q^{i_k} x^{i_k}$ and applying this inequality, we find

$$\sum_{i_k=0}^{d_k} \left| b(k)_{i_1 \cdots i_k} \right| Q^{i_k} \leqq 2^{d_k} \sum_{i_k=0}^{d_k} \left| a(k)_{i_1 \cdots i_k} \right| Q^{i_k}.$$

This we multiply by $Q^{i_1 + \cdots + i_{k-1}}$ and sum over all allowed $i_1, \cdots, i_{k-1}$. The result so obtained is that[3]

(6) $$L_Q(g_k) \leqq 2^{d_k} L_Q(f_k).$$

Successive application of (5), (6), and the easily obtained formula

(7) $$L_Q(f_{k-1}) \leqq L_Q(g_k)$$

yields

(8) $$L_Q(g_k) \leqq 2^{d_k + \cdots + d_n} Q^{d_1 + \cdots + d_n} L(f) \qquad (k = n, \cdots, 1).$$

Let now $\epsilon_1, \cdots, \epsilon_n$ be real numbers, and assume that the former inequalities for q and p_k are replaced by the stronger formulae

(9) $$|q| \leqq Q, \quad |p_k| \leqq Q, \quad |p_k + \epsilon_k| \leqq Q \qquad (k = 1, \cdots, n).$$

It follows then from the mean value theorem that

$$\begin{aligned} \Delta_k \equiv\ & g_{k+1}(p_1 + \epsilon_1, \cdots, p_k + \epsilon_k, p_{k+1} + \epsilon_{k+1}) \\ & - g_{k+1}(p_1 + \epsilon_1, \cdots, p_k + \epsilon_k, p_{k+1}) \\ =\ & \epsilon_{k+1} \frac{\partial}{\partial x_{k+1}} g_{k+1}(p_1 + \epsilon_1, \cdots, p_k + \epsilon_k, x_{k+1}) \Big|_{x_{k+1} = p_{k+1} + \theta \epsilon_{k+1}}, \end{aligned}$$

[3] A reader not familiar with Mahler's inequality should have no difficulty proving $L_Q(g_k) \leqq c_1(f) L_Q(f_k)$. This yields a theorem where the expression in brackets in (2) is replaced by $c_2(f)$, and it yields our condition for independence.

where θ is a certain constant satisfying $0<\theta<1$, so that also $|x_{k+1}| \leqq Q$. Thus we find that

$$|\Delta_k| \leqq |\epsilon_{k+1}| d_{k+1}Q^{-1}L_Q(g_{k+1})$$

and hence by (8),

$$|\Delta_k| \leqq |\epsilon_{k+1}| Q^{d_1+\cdots+d_n-1}d_{k+1}2^{d_{k+1}+\cdots+d_n}L(f). \tag{10}$$

Assume now that $|\epsilon_k| \leqq 1$ $(k=1, \cdots, n)$ and that

$$|\epsilon_k| \leqq |\epsilon_1|^{d_1} \cdots |\epsilon_{k-1}|^{d_{k-1}}Q^{1-d_1-\cdots-d_n}[d_k 2^{d_k+\cdots+d_n+k}L(f)]^{-1} \quad (k = 1, \cdots, n). \tag{11}$$

We are going to show

$$|f_k(p_1+\epsilon_1, \cdots, p_k+\epsilon_k)| \geqq |\epsilon_1|^{d_1} \cdots |\epsilon_k|^{d_k}2^{-k} \quad (k = 0, 1, \cdots, n). \tag{12}$$

The formula is true for $k=0$ since $|f_0| \geqq 1$. If it is true for some $k < n$, then by (10) and (11),

$$|g_{k+1}(p_1+\epsilon_1, \cdots, p_{k+1}+\epsilon_{k+1}) - f_k(p_1+\epsilon_1, \cdots, p_k+\epsilon_k)| \leqq 2^{-1}|f_k(p_1+\epsilon_1, \cdots, p_k+\epsilon_k)|,$$

hence

$$\begin{aligned} |f_{k+1}(p_1+\epsilon_1, \cdots, p_{k+1}+\epsilon_{k+1})| &= |\epsilon_{k+1}|^{\delta_{k+1}}|g_{k+1}(p_1+\epsilon_1, \cdots, p_{k+1}+\epsilon_{k+1})| \\ &\geqq 2^{-k-1}|\epsilon_1|^{d_1} \cdots |\epsilon_k|^{d_k}|\epsilon_{k+1}|^{\delta_{k+1}}, \end{aligned}$$

and (12) is true for $k+1$.

The proof of the theorem is now immediate. Assume (1) and (2) to be satisfied. Put $\xi_k q = p_k + \epsilon_k$ $(k=1, \cdots, n)$, where p_k is integral and $|\epsilon_k| = \|\xi_k q\|$. Then (1) yields (9) because Q and the p_k's are integral, and (2) gives (11). We obtain

$$\begin{aligned} |q^{d_1+\cdots+d_n}f(\xi_1, \cdots, \xi_n)| &= |f_n(p_1+\epsilon_1, \cdots, p_n+\epsilon_n)| \\ &\geqq |\epsilon_1|^{d_1} \cdots |\epsilon_n|^{d_n}2^{-n}, \end{aligned}$$

thereby proving our theorem.

I am grateful to Professor Mahler for simplifying my original proof.

References

1. J. Liouville, *Sur les classes très etendus de quantites dont la valeur n'est algebriques, etc.*, C. R. Acad. Sci. Paris **18** (1844), 883–885, 910–911.

2. K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika **7** (1960), 98–100.

3. J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Ann. of Math. **99** (1928), 134–141.

4. Th. Schneider, *Einführung in die transzendenten Zahlen*, Springer, Berlin, 1957.

Columbia University