INERTIA THEOREMS FOR MATRICES: THE SEMI-DEFINITE CASE

BY DAVID CARLSON AND HANS SCHNEIDER[1]

Communicated by A. S. Householder, June 4, 1962

1. The *inertia* of a square matrix A with complex elements is defined to be the integer triple In $A = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ $\{\nu(A)\}$ equals the number of eigenvalues in the open right $\{$left$\}$ half plane, and $\delta(A)$ equals the number of eigenvalues on the imaginary axis. The best known classical inertia theorem is that of Sylvester: If $P>0$ (positive definite) and H is Hermitian, then In $PH=$ In H. Less well known is Lyapunov's theorem [**2**]: There exists a $P>0$ such that $\mathcal{R}(AP)=\frac{1}{2}(AP+PA^*)>0$ if and only if In $A=(n, 0, 0)$. Both classical theorems are contained in a generalization (Taussky [**4**], Ostrowski-Schneider [**3**]) which we shall call the

MAIN INERTIA THEOREM. *For a given A, there exists a Hermitian H such that $\mathcal{R}(AH)>0$ if and only if $\delta(A)=0$. If $\mathcal{R}(AH)>0$, then* In $A=$ In H.

2.1. In this note we consider the case $\mathcal{R}(AH)\geqq 0$ which is far more complicated than the case $\mathcal{R}(AH)>0$. We do not here solve the problem of all the possible relations of In H to In A, except under additional assumptions.

THEOREM 1. *Let A be a given matrix for which all elementary divisors of imaginary eigenvalues are linear. If H is a Hermitian matrix such that $\mathcal{R}(AH)\geqq 0$, then $\pi(H)=\pi$, $\nu(H)=\nu$ satisfy*

$$(1) \qquad \pi \leqq \pi(A)+\delta(A), \qquad \nu \leqq \nu(A)+\delta(A),$$

respectively, and

$$(2) \qquad \text{rank } \mathcal{R}(AH) \leqq \pi(A)+\nu(A).$$

Further, for any triple (π, ν, δ) for which $\pi+\nu+\delta=n$, and π, ν satisfy (1), there exists an H for which $\mathcal{R}(AH)\geqq 0$ and In $H=(\pi, \nu, \delta)$. *Thus (1) is in a sense the best possible inequality.*

A more precise result may be proved if rank $\mathcal{R}(AH)=\pi(A)+\nu(A)$.

2.2. Theorem 2 concerns a matrix consisting of just one Jordan

[1] The research of the authors was supported by the National Science Foundation under grant No. NSF G-19052 and by the United States Army under Contract No. DA-11-022-ORD-2059.

block with one imaginary root. Its proof is largely computational. For assertion (4) (below) we use Cauchy's theorem on the separation of eigenvalues of a Hermitian matrix by the eigenvalues of a principal minor.

THEOREM 2. *Let $A=\alpha I+U$, where α is pure imaginary and U is the matrix with* 1 *in the first superdiagonal and* 0 *elsewhere. If H is Hermitian of* rank r *and $K=\mathcal{R}(AH)\geqq 0$ is of* rank s, *then*

$$2s \leqq r, \tag{3}$$

and for $\pi(H)=\pi$, $\nu(H)=\nu$,

$$|\pi-\nu| \leqq 1, \tag{4}$$

$$h_{ij}=0 \qquad \text{if } i+j>r+1, \tag{5}$$

$$k_{ij}=0 \qquad \text{if } i>r/2. \tag{6}$$

Again, the inequalities (3) *and* (4) *are best possible, in the sense that if r, s, π, ν, with $\pi+\nu=r$, are non-negative integers satisfying* (3) *and* (4) *then we can find an H such that $\mathcal{R}(AH)\geqq 0$, and $r=\text{rank } H$, $s=\text{rank } \mathcal{R}(AH)$, $\pi=\pi(H)$ and $\nu=\nu(H)$.*

As a corollary of Theorem 2 we obtain a general existence theorem:

COROLLARY. *For any matrix A, there exists a nonsingular Hermitian H such that $\mathcal{R}(AH)\geqq 0$.*

In the special case that all elementary divisors of imaginary roots are linear, this result is known; cf. Givens [1].

2.3. THEOREM 3. *Let A be a given matrix. If $H\geqq 0$ and $\mathcal{R}(AH)\geqq 0$, then*

$$\text{rank } H \leqq \pi(A)+p(A), \tag{7}$$

where $p(A)$ is the number of elementary divisors of imaginary roots. The inequality (7) *is best possible.*

COROLLARY 1. *For a given matrix A, there exists an $H>0$ for which $\mathcal{R}(AH)\geqq 0$ if and only if*

$$\nu(A)=0, \tag{8}$$

(9) *all elementary divisors of imaginary eigenvalues of A (if any) are linear.*

COROLLARY 2. *If $\mathcal{R}(A)\geqq 0$ and $H>0$ then all elementary divisors of imaginary eigenvalues of AH are linear.*

When $H=I$, Corollary 2 reduces to part of Theorem 2 of [3].

3.1. The proof of the Main Inertia Theorem hinges on the following lemma: If $\mathfrak{R}(AH)>0$, then H is nonsingular. In this section we shall obtain a generalization of the Main Theorem by considering matrices with fixed null-space $\mathfrak{N}$. By $\mathfrak{N}(A)$ we shall denote the null-space of $A\,(x\in\mathfrak{N}(A): Ax=0)$ and $\mathfrak{N}^{\perp}$ will be the orthogonal complement of $\mathfrak{N}(x\in\mathfrak{N}^{\perp}: y^*x=0$ for all $y\in\mathfrak{N})$. Our results depend on the easily proved Theorem 4 which takes the place of the lemma quoted above.

We define $\operatorname{In} A\leqq \operatorname{In} B$ if $\pi(A)\leqq\pi(B)$ and $\nu(A)\leqq\nu(B)$ (A, B need not be of the same order), and $\operatorname{In} A=\operatorname{In} B$ if $\pi(A)=\pi(B)$ and $\nu(A)=\nu(B)$.

THEOREM 4. *If* $\mathfrak{R}(AH)\geqq 0$ *then*

$$\mathfrak{N}(\mathfrak{R}(AH)) \supseteq \mathfrak{N}(H), \tag{10}$$

$$A\mathfrak{N}(H)^{\perp} \subseteq \mathfrak{N}(H)^{\perp}, \tag{11}$$

$$\operatorname{In}(A \mid \mathfrak{N}(H)^{\perp}) \leqq \operatorname{In} H. \tag{12}$$

Here $A\mid\mathfrak{N}(H)^{\perp}$ is the restriction of A to $\mathfrak{N}(H)^{\perp}$.

As an immediate corollary to the proposition we have

COROLLARY. *If* $\mathfrak{R}(AH)\geqq 0$ *and* $\operatorname{In}(A^*\mid\mathfrak{N}(H))=(0,0,\delta)$ *then*

$$\operatorname{In} A = \operatorname{In}(A \mid \mathfrak{N}(H)^{\perp}) \leqq \operatorname{In} H.$$

In particular if $\mathfrak{R}(AH)\geqq 0$ *and H is nonsingular, then* $\operatorname{In} A\leqq\operatorname{In} H$.

3.2. It is interesting to note that in our next theorem, the inequalities will go in the opposite direction. This theorem reduces to the Main Inertia Theorem when $\mathfrak{N}=(0)$.

THEOREM 5. *Let $\mathfrak{N}$ be a subspace of V. There exists a Hermitian H such that*

$$\mathfrak{R}(AH) \geqq 0, \tag{13}$$

and

$$\mathfrak{N}(\mathfrak{R}(AH)) = \mathfrak{N}(H) = \mathfrak{N}, \tag{14}$$

if and only if

$$A\mathfrak{N}^{\perp} \subseteq \mathfrak{N}^{\perp} \tag{15}$$

and

$$\delta(A \mid \mathfrak{N}^{\perp}) = 0. \tag{16}$$

If (13) *and* (14) *hold, then*

$$\operatorname{In} H = \operatorname{In}(A \mid \mathfrak{N}^{\perp}) \leqq \operatorname{In} A.$$

COROLLARY 1. *Let A and $\mathfrak{N}$ satisfy conditions* (15) *and* (16). *If* $\mathfrak{R}(AH)\geqq 0$ *and* $\mathfrak{N}(H)\supseteq\mathfrak{N}$, *then* $\text{In } H\leqq\text{In}(A\,|\,\mathfrak{N}^\perp)\leqq\text{In } A$.

COROLLARY 2. *If* $\delta(A)=0$ *and* $\mathfrak{R}(AH)\geqq 0$, *then* $\text{In } H\leqq\text{In } A$. *If, in addition,* $\delta(H)=0$ (*i.e.,* H *is nonsingular*), *then* $\text{In } H=\text{In } A$.

COROLLARY 3. *If* $\mathfrak{R}(AH)\geqq 0$ *and* $\text{rank }\mathfrak{R}(AH)=\text{rank } H=\pi(A)+\nu(A)$, *then, again,* $\text{In } H=\text{In } A$.

3.3. Suppose the conditions of Theorem 5 are fulfilled and there exists a K such that $\mathfrak{R}(AK)\geqq 0$, and $\mathfrak{N}(\mathfrak{R}(AK))=\mathfrak{N}(K)=\mathfrak{N}$, A and $\mathfrak{N}$ being given. When does every H satisfying $\mathfrak{N}(\mathfrak{R}(AH))=\mathfrak{N}$ (and not necessarily satisfying $\mathfrak{R}(AH)\geqq 0$) also satisfy $\mathfrak{N}(H)=\mathfrak{N}$? For $\mathfrak{N}=(0)$, the question is: When does $\mathfrak{R}(AH)=0$ imply $H=0$?. The conditions for this are well-known (Corollary below). Thus our Theorem 6 is a generalization of the known Corollary 6.

We require the following definition. If A and B are square matrices (possibly of different orders), we let

$$T(A, B) = \prod_{i,j} (\alpha_i + \beta_j)$$

the product being taken over all pairs of eigenvalues (α_i, β_j) of A and B, and for the sake of convenience we write $T(A)=T(A, A^*)$. If A is the empty matrix (an operator on a 0-dimensional space), certain consistency conditions force us to take $T(A, B)=1$.

THEOREM 6. *Let $\mathfrak{N}$ be a subspace of V, and A a matrix for which* $A\mathfrak{N}^\perp\subseteq\mathfrak{N}^\perp$. *If*

$$T(A\,|\,\mathfrak{N}^\perp, A^*\,|\,\mathfrak{N})\cdot T(A^*\,|\,\mathfrak{N}) \neq 0 \tag{17}$$

then $\mathfrak{N}(\mathfrak{R}(AH))\supseteq\mathfrak{N}$ *implies* $\mathfrak{N}(H)\supseteq\mathfrak{N}$. *Conversely, if*

$$T(A\,|\,\mathfrak{N}^\perp, A^*\,|\,\mathfrak{N})\cdot T(A^*\,|\,\mathfrak{N}) = 0 \tag{18}$$

then there exists a Hermitian H such that $\mathfrak{N}(\mathfrak{R}(AH))\supseteq\mathfrak{N}$ *but* $\mathfrak{N}(H)\not\supseteq\mathfrak{N}$.

COROLLARY 1. *There exists a nonzero H such that* $\mathfrak{R}(AH)=0$ *if and only if* $T(A)=0$.

COROLLARY 2. *Let* $\mathfrak{R}(AH)\geqq 0$ *and let* $\mathfrak{N}=\mathfrak{N}(\mathfrak{R}(AH))$. *If* $A\mathfrak{N}^\perp\subseteq\mathfrak{N}^\perp$ *and* (17) *holds then* $\mathfrak{N}(\mathfrak{R}(AH))=\mathfrak{N}(H)$.

COROLLARY 3. *Let* $\mathfrak{R}(AK)\geqq 0$ *and* $\mathfrak{N}=\mathfrak{N}(K)=\mathfrak{N}(\mathfrak{R}(AK))$. *If* (17) *holds, then* $\mathfrak{R}(AH)\geqq 0$ *and* $\mathfrak{N}(\mathfrak{R}(AH))=\mathfrak{N}$ *implies that* $\mathfrak{N}(H)=\mathfrak{N}$. *Conversely if* (18) *holds, then there exists a Hermitian H such that* $\mathfrak{R}(AH)\geqq 0$ *and* $\mathfrak{N}(\mathfrak{R}(AH))=\mathfrak{N}$ *but* $\mathfrak{N}(H)$ *is properly contained in* $\mathfrak{N}$.

4. As in [3], the matrix A is called H-stable if, for Hermitian matrices H, In $AH=(n, 0, 0)$ if and only if $H>0$. A necessary and sufficient condition for H-stability was found in [3], Theorem 4. However, this condition does not greatly facilitate the determination of H-stability for a given matrix A. Our Theorem 7 below provides an effective test for H-stability. The only candidates are nonsingular A with $\mathcal{R}(A)\geqq 0$, and thus we need merely diagonalize $\mathcal{R}(A)$ and examine the transform of $\mathcal{I}(A)=(1/2i)(A-A^*)$.

THEOREM 7. *Let A be a nonsingular matrix with $\mathcal{R}(A)\geqq 0$, and let $k=\max_{H>0}\delta(AH)$. Let S be any nonsingular matrix for which $S^*AS=A'=P+iQ$, where $P=P_{11}\oplus 0$ and Q are Hermitian, and $P_{11}>0$. If Q is partitioned conformably with P, then rank $Q_{22}=k$. In particular, A is H-stable if and only if $Q_{22}=0$.*

COROLLARY. *If A is an H-stable matrix of order n, then rank $\mathcal{R}(A)\geqq n/2$.*

REFERENCES

1. W. Givens, *Elementary divisors and some properties of the Lyapunov mapping $X\rightarrow AX+XA^*$*, Argonne National Laboratory Report ANL-6456, 1961.

2. A. Lyapunov, *Problême géneral de la stabilité du mouvement*, Comm. Soc. Math. Kharkov (1892); Annals of Mathematics Studies, No. 17, Princeton Univ. Press, Princeton, N. J., 1947.

3. A. M. Ostrowski and H. Schneider, *Some theorems on the inertia of general matrices*, J. Math. Anal. Appl. **4** (1962), 72–84.

4. O. Taussky, *A generalization of a theorem by Lyapunov*, J. Soc. Indust. Appl. Math. **9** (1961), 640–643.

UNIVERSITY OF WISCONSIN