COVERS AND PACKINGS IN A FAMILY OF SETS[1]

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1. For a finite set S of elements and a family F of subsets of S, a *cover* C of S from F is a subfamily $C \subset F$ such that $\bigcup(C) = S$. A cover C is called minimum if its cardinality $|C|$ is as small as possible. A *packing* D in F is a subfamily of F whose members are disjoint. It is called maximum if its cardinality $|D|$ is as large as possible. Theorem 1 here is relevant to the task of finding minimum covers. Theorem 2, which follows easily from Theorem 1, is the analogous result on maximum packings. Finally, Theorem 3 extends the foregoing to "α-covers."

Minimum covers are equivalent to solutions of the following integer program: Minimize $\sum x_i$ by a vector $x = (x_1, \cdots, x_n)'$ of zeroes and ones for which $Ax \geqq \bar{1} = (1, \cdots, 1)'$. Here A is the zero-one incidence matrix of members of F (columns) versus members of S (rows). Where $\bar{1}$ is replaced by a vector α of arbitrary positive integers, Fulkerson and Ryser call min $\sum x_i$ the α-width of A. In [3] they find a lower bound for the α-width of zero-one matrices A with given row and column sums.

By analogy with α-width, an *α-cover* of S from F, where α is a vector whose components correspond to the members of S, is a subfamily of F of which at least α_i members contain $y_i \in S$. A β-packing in F is a subfamily of F of which at most β_i members contain $y_i \in S$. Where $\alpha_i + \beta_i$ is the number of members of F which contain $y_i \in S$, the complement in F of an α-cover is a β-packing, and conversely.

The Berge-Norman-Rabin theorem [1], which concerns α-covers where each member of F contains exactly two elements, is based on "alternating paths," invented in 1891 by Peterson and used frequently to prove theorems about linear graphs. Theorem 1 generalizes the N-R instance [5] of the B-N-R theorem, where $\alpha = \bar{1}$, by extending the notion of alternating paths to "alternating trees." Analogously, Theorem 2 includes the Berge theorem [2] on maximum matchings (packings of edges) in a graph. By introducing "self-intersecting trees," Theorem 3 generalizes the B-N-R theorem. Its proof, essentially the same as the proof of Theorem 1, is not given.

Although present knowledge on practical algorithms is quite limited, in theory at least minimum covering and related program-

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ming problems have many combinatorial applications. One well known in electrical engineering is the search for a "simplest" normal disjunctive form of a Boolean function (cf. [6]). To cite another example, if block designs [4] exist for a given set of parameters including $\lambda = 1$, they correspond precisely to the minimum covers (maximum packings) from a certain family of sets.

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2. Represent a finite set S of elements and a family F of its subsets by a linear graph B whose vertices are the members of S and the members of F so that an S vertex, y, is joined (by an edge) in B to an F vertex, x, if and only if $y \in x$. These are the only edges in B. A cover C is a subset of the F vertices at least one of which is joined to each S vertex. Denote $F-C$ by $\bar{C}$; thus the vertices of B fall into three disjoint classes, S, C and $\bar{C}$.

Here it is convenient to define *tree* inductively: A graph T is a tree either if it consists of a single vertex or else if it consists of two disjoint trees, T_1 and T_2, together with an edge joining a vertex in T_1 to a vertex in T_2. We say graph $G' = G - e$ is obtained from graph G by deleting edge e, if G consists of G' and an edge e not in G' which joins a pair of vertices of G'. We use the fact that the graph obtained by deleting any edge from a tree consists of exactly two disjoint trees.

THEOREM 1. *A cover $C \subset F$ of S is* not *minimum if and only if B contains as a subgraph a tree T such that*

(1) *each $S \cap T$ vertex meets exactly two edges of T, one joining it to a $C \cap T$ vertex and the other joining it to a $\bar{C} \cap T$ vertex;*

(2) *each $\bar{C} \cap T$ vertex meets exactly two edges of T;*

(3) $C' = (C - T) \cup (\bar{C} \cap T)$ *is a cover.*

PROOF OF SUFFICIENCY. Each $S \cap T$ vertex together with its two edges in T may be regarded as a single edge, which we call a *bi-edge*, joining a $C \cap T$ vertex to a $\bar{C} \cap T$ vertex. Each $\bar{C} \cap T$ vertex together with its two bi-edges in T may be regarded as a single bi-bi-edge joining two $C \cap T$ vertices. The bi-bi-edges, in 1-1 correspondence with the $\bar{C} \cap T$ vertices, form a tree whose vertices are those of $C \cap T$. It is well known and obvious that the number of vertices of a tree is one greater than the number of its edges. Therefore $|C'| < |C|$.

3. LEMMA. *If cover C is not minimum then B contains a tree obeying* (1), (3) *and*

(2′) $|\bar{C} \cap T| < |C \cap T|$, *i.e.*, $|C'| < |C|$.

The lemma holds as well, if cardinality is replaced by any real additive set-function on the subfamilies of F. The proof would be the same as below if cardinality were replaced in *some* places there by function-value.

PROOF OF THE LEMMA. Since $|C|$ is not minimum, there is a cover M of smaller cardinality. Among these covers M choose one for which $|(C-M)\cup(M-C)|$ is minimum. The graph B does contain trees T obeying condition (1) as well as

(2″) $C\cap T\subseteq C-M$ and $\overline{C}\cap T\subseteq M-C$, i.e., $C\cap T=\overline{M}\cap T$ (where $\overline{M}=F-M$) and $\overline{C}\cap T=M\cap T$.

A single vertex of the nonempty set $(C-M)\cup(M-C)$ is such a tree. Among such trees T, choose one with a maximum number of vertices.

We first show that T obeys condition (3). Any $S-T$ vertex, y, not joined in B to a $C-T$ vertex must be joined to a $C\cap T$ vertex, x, since C is a cover. Since M is a cover, y must also be joined in B to some M vertex, z. If z were not in T (thus not in C by the hypothesis on y) then T could be enlarged to a tree still obeying (1) and (2″) by adjoining the bi-edge from x to y to z. This contradicts the maximality of T, so z is in T, i.e., z is in $M\cap T=\overline{C}\cap T$. So any $S-T$ vertex y not joined to a $C-T$ vertex is joined to a $\overline{C}\cap T$ vertex. Since every $S\cap T$ vertex is joined to a $\overline{C}\cap T$ vertex by condition (1), C' of condition (3) is indeed a cover.

Similarly we can show that $M'=(M-T)\cup(\overline{M}\cap T)$ is a cover. If $\overline{M}\cap T=C\cap T$ were no larger than $M\cap T=\overline{C}\cap T$, then M' as well as M would be a smaller cover than C. This contradicts the choice of M since $(C-M')\cup(M'-C)=[(C-M)\cup(M-C)]-T$ is smaller than $(C-M)\cup(M-C)$. Therefore (2′), $|\overline{C}\cap T|<|C\cap T|$, and so the lemma is proved. In fact, it follows that $C\cap T=C-M$, $\overline{C}\cap T=M-C$, and $C'=M$.

4. PROOF OF THEOREM 1. For a vertex x in a tree T, let $T(x)$ denote the number of edges in T which meet x. In a tree T obeying (1) and (2′) of the lemma, either (2) holds, i.e., $T(x)=2$ for all $x\in\overline{C}\cap T$, or else there is an $x\in\overline{C}\cap T$ such that $T(x)>2$. For suppose neither holds. By removing from T each $x_0\in\overline{C}\cap T$ for which $T(x_0)=1$ together with the bi-edge joining x_0 to a $C\cap T$ vertex, one obtains a tree T' obeying (2). As observed in the sufficiency proof, (1) and (2) imply that T' has only one more C vertex than $\overline{C}$ vertices. Since T' was obtained by deleting $\overline{C}\cap T$ vertices, we have a contradiction to (2′).

It also follows from (2′) that each $x\in\overline{C}\cap T$ with $T(x)>2$ meets at least one bi-edge e of T such that the component, R_1, of $T-e$ containing x has more C vertices than $\overline{C}$ vertices. The other component of $T-e$ will be called R_2. The desired e can be obtained by choosing one for which $|C\cap R_2|<|\bar{C}\cap R_2|$ if possible, and otherwise by choosing any bi-edge to x.

Assuming C is not minimum, there exist by the lemma trees T obeying (1), (2′) and (3). Let $f(T)=\sum_x (T(x)-2)$, where $x\in\bar{C}\cap T$ and $T(x)>1$. We want to show that at least one of these trees satisfies (2)—that is, $f(T)=0$.

Assume to the contrary that f is positive for all of the trees. Then from among those that minimize f we can choose one, still calling it T, which contains an e such that R_2, as defined above, contains a minimum number of vertices. We shall obtain a contradiction from T by deleting the e which joins R_1 and R_2 and then reconnecting these components by a new bi-edge e_0 from B to obtain a new T^* obeying (1), (2′) and (3), such that $f(T^*)=f(T)$, and containing an R_2^* smaller than R_2.

Component R_1 is a tree with properties (1) and (2′) and such that $f(R_1)<f(T)$. Therefore, by the choice of T, R_1 cannot have property (3)—that is, some S vertex y_0 is joined neither to a $C-R_1$ vertex nor to a $\bar{C}\cap R_1$ vertex. This y_0 will be the S vertex of the bi-edge e_0. For T^* to be a tree obeying (1), y_0 must not lie in either R_1 or R_2, which is true since each S vertex of R_1 is joined to a $\bar{C}\cap R_1$ vertex and each S vertex of R_2 is joined to a $C-R_1$ vertex.

Observe that y_0 is joined (in B) to a $z_0\in C\cap R_1$ because C is a cover, and also to an $x_0\in\bar{C}\cap R_2$ because $(C-T)\cup(\bar{C}\cap T)$ is a cover. Thus e_0 can be taken as the bi-edge joining x_0 to y_0 to z_0. Tree $T^*=R_1\cup R_2\cup e_0$ obeys (1), (2′) and (3).

Recall that x is the $\bar{C}$ vertex of bi-edge e. Since e_0 has its C vertex in R_1 and its $\bar{C}$ vertex in R_2, the reverse of the situation for e, we have $x_0\neq x$. Therefore $T^*(x)=T(x)-1$ and $T^*(x_0)=T(x_0)+1$. For all other vertices x_1 in $\bar{C}\cap T^*=\bar{C}\cap T$, $T^*(x_1)=T(x_1)$.

If $T^*(x_0)=2$ then x_0 contributes neither to $f(T)$ nor to $f(T^*)$, whereas, by definition, x does contribute to $f(T)$. Thus we have $f(T^*)=f(T)-1$, contradicting the minimality of $f(T)$. Therefore $T^*(x_0)>2$, and $f(T^*)=f(T)$.

The choice of T and R_2 also requires that T^* contain no R_2^*, defined like R_2 in T, with fewer vertices than R_2. However, let $e_1\neq e_0$ be a bi-edge of T^* which meets x_0. If there are any, let e_1 be one such that R_2^*, the component of T^*-e_1 not containing x_0, contains no more C vertices than $\bar{C}$ vertices. Since $T^*(x_0)>2$, and $|C\cap T^*|>|\bar{C}\cap T^*|$, the other component R_1^* of T^*-e_1 has more C than $\bar{C}$ vertices. Since $T^*(x_0)>2$, R_2^* is a proper subtree of R_2. Therefore the minimality of R_2 is contradicted and the theorem is proved.

5. When each set in F consists of exactly two elements from S, each F vertex in graph B meets exactly two edges and hence the tree is a path. For this case all except the lemma of the necessity proof is superfluous. It is convenient in this case to regard (S, F) as a graph G, whose edges are the members of F and whose vertices are the members of S.

COROLLARY 1 (NORMAN AND RABIN [5]). *If C, a family of edges in G which meets each of the vertices in G, is not one of minimum cardinality then there exists in G a path P (simple except possibly at vertices which are ends of the path) such that* (1) *edges of P are alternately in C and $\overline{C}$, and* (2) *$C' = (C-P) \cup (\overline{C} \cap P)$ is a smaller cover than C.*

Like the lemma, Corollary 1 holds when cardinality is replaced by the value of an arbitrary real additive function on the sets of edges in G.

6. Another important special case of Theorem 1 is when each element of S is contained in exactly two members of F. This is equivalent to minimum covering the edges of a graph G by vertices of G; the edges are the members of S (the bi-edges of B) and the vertices are the members of F. For this case, iff C is a cover, $D = \overline{C}$ is a packing. Graphically speaking, iff C is a set of vertices which meets every edge at least once, then the vertices not in C meet each edge at most once since an edge has two ends. A set D of vertices in G such that no two are joined by an edge is called an *internally stable set* or a *packing* in G. Thus for a graph G with a nonmaximum packing D, Theorem 1 yields a COROLLARY 2 asserting the existence in G of a tree each of whose edges joins a D vertex to a $\overline{D} = F - D$ vertex and which obeys properties (2) and (3) of T in Theorem 2 below.

THEOREM 2. *A packing $D \subset F$ is not maximum if and only if B contains as a subgraph a tree T such that*

(1) *each $S \cap T$ vertex meets exactly two edges of T, one joining it to a $D \cap T$ vertex and the other joining it to a $\overline{D} \cap T$ vertex;*

(2) *each $D \cap T$ vertex meets exactly two edges of T;*

(3) *$D' = (D - T) \cup (\overline{D} \cap T)$ is a packing.*

Corollary 2 leads almost immediately to Theorem 2, the packing-analog of Theorem 1. Graph B is the same as before. Consider also a graph G whose vertices are the members of F and such that two vertices are joined by an edge if and only if, as sets, their intersection is nonempty. The packings in F are obviously identical to the packings in G.

For a nonmaximum packing D, let T_0 be one of the trees in G whose existence is asserted by Corollary 2. Construct a subgraph T of B as follows. The D vertices and the $\bar{D}$ vertices of T are the same as the vertices of T_0. For each edge e in T_0, joining vertices x_1 and x_2, let y be *one* of the elements common to sets x_1 and x_2. Let T contain S vertex y of B and the two edges of B which join y to vertices x_1 and x_2. It is easy to show that graph T is a tree in B with the properties described in Theorem 2.

7. In terms of the graph B, an α-cover C is a subset of the F vertices such that each S vertex, y_i, is joined to at least α_i members of C. A β-packing D is a subset of the F vertices such that each S vertex, y_i, is joined to at most β_i members of D.

Let a graph G, together with a graph G' and a mapping of G' onto G which is bi-unique with respect to edges and which preserves edge-vertex incidences, be called a *self-intersecting copy* of G'. A subgraph T of B we call an *α-tree* if it is a self-intersecting copy of a tree T' whose vertices are partitioned into classes F' and S' so that each $F \cap T$ vertex is the image of exactly one vertex, which lies in F', and each $S \cap T$ vertex, y_i, is the image of at most α_i vertices, all in S'. For subsets C and $\bar{C}$ of F, denote the inverse images of $C \cap T$ and $\bar{C} \cap T$ by $C' \cap T'$ and $\bar{C}' \cap T'$, respectively.

THEOREM 3. *In Theorem* 1 *replace "cover" by "α-cover," "tree" by "α-tree," and S, C, $\bar{C}$, and T in condition* (1) *by S', C', $\bar{C}'$, and T'.*

Let $\alpha_i + \beta_i$ be the number of F vertices joined in B to S vertex y_i. Because $\bar{C}$ is a β-packing, it follows automatically from condition (1) of Theorem 3 that T is a β-tree as well as an α-tree. Thus Theorem 3 generalizes Theorem 2 as well as Theorem 1.

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