NOTE ON Γ*-SEMIGROUPS

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The system $L(S)$ of all nonvoid subsemigroups of a semigroup S is generally a semilattice[1] with respect to the inclusion relation. $L(S)$ is called the subsemigroup semilattice of S. In the previous paper [1] we determined all the Γ-semigroups,[2] i.e., the semigroups whose subsemigroup semilattices are chains. In detail, all the types of Γ-semigroups are

(1.1) cyclic groups $G(p^n)$ of order of prime power,

(1.2) quasi-cyclic groups $G(p^\infty)$,

(1.3) unipotent semigroups generated by d with each of the following defining relations:

(1.3.1) $d^2=d^3$,

(1.3.2) $d^3=d^4$,

(1.3.3) $d^2=d^{p^m+2}$, p prime,

(1.3.4) $d^3=d^{p^m+3}$, p prime $\neq 2$.

In the present note, we shall define Γ*-semigroups as generalizations of Γ-semigroups and shall report the structure of Γ*-semigroups except for a part of infinite Γ*-groups. The proof will be omitted here but will be given elsewhere.[3]

DEFINITION. A semigroup S is called a Γ*-semigroup if every subsemigroup different from S is a Γ-semigroup.

S is a Γ*-semigroup if and only if $L(S)$ is a semilattice satisfying: Any subset which contains the greatest element is a subsemilattice. A semilattice of this kind is called a C_0-semilattice. Obviously all the semigroups of order 2 are Γ*-semigroups, and a homomorphic image of a Γ*-semigroup is also a Γ*-semigroup.

LEMMA 1. *Every element of a Γ*-semigroup is of finite order, that is, for any element x there is an idempotent e and a positive integer n such that $x^n=e$.*

LEMMA 2. *A Γ*-semigroup of order >2 is unipotent. (i.e., an idempotent element is unique).*

Generally a unipotent semigroup any element of which is of finite

[1] By a semilattice we mean a partially ordered set in which there is a join of two elements.

[2] In [1] we called them Γ-monoids.

[3] *Semigroups and their subsemigroups semilattices,* to appear.

order is determined by a group and a Z-semigroup (i.e., a unipotent semigroup with zero) [**2**; **3**]. By Lemmas 1 and 2, we can make the discussion proceed to Γ^*-Z-semigroups, Γ^*-groups, and then to the general cases.

THEOREM 1. *Any Γ^*-Z-semigroup is of order $\leqq 4$. All the types of Γ^*-Z-semigroups other than Γ-semigroups are listed as follows:*

(2.1) $\{0, a, b\}$ *of order* 3 *where* $xy=0$ *for all* x, y,

(2.2) $\{0, a, b, c\}$ *of order* 4 *defined as*

(2.2.1) $b^2=c^2=a$ *and other products* $=0$.

(2.2.2) $b^2=cb=c^2=a$ *and other products* $=0$.

(2.2.3) $b^2=c^2=bc=cb=a$, *and other products* $=0$.

As far as the Γ^*-groups are concerned, we shall limit ourselves to the case of Γ^*-groups which are properly homomorphic to Γ-groups.

We can prove that any Γ^*-group which is properly homomorphic to a Γ-group has a normal subgroup of index of a prime number. Making use of the theory of finite groups [**4**; **5**; **6**], we have

THEOREM 2. *Any Γ^*-group, which is not a Γ-group and is homomorphic to a Γ-group of order >1, has one of the following types.*

(3.1) *The groups of order pq where p and q are different primes. There are two types* (3.1).

(3.2) *The elementary abelian group*: $G(p)\times G(p)$.

(3.3) *The generalized quaternion group of order* 8.

Incidentally a finite Γ^*-group, which is not a Γ-group, is homomorphic to a Γ-group; a commutative Γ^*-group which is not a Γ-group is the direct product of two groups of prime order. Consequently we see that the result of Theorem 2 includes the cases where a Γ^*-group is homomorphic to a nontrivial finite group or a commutative group. However the problem of determination of the remaining case is still open.

Next, let S be a unipotent Γ^*-semigroup which is neither a group nor a Z-semigroup. Then we can prove that S must be finite. The kernel (i.e., the least ideal) of S is of type $G(p^n)$, and the difference semigroup D of S modulo $G(p^n)$, due to Rees [**7**] is a Z-semigroup which has one of the types (1.3.1), (2.1), (2.2.1), (2.2.2), (2.2.3).

Let e be the unique idempotent of S, and let d be a generator of D. $G(p^{n-1})$ will denote the subgroup of order p^{n-1} of $G(p^n)$.

THEOREM 3. *When $G(p^n)$ is given, we can determine all the unipotent Γ^*-semigroups, non Γ-semigroups, whose kernel is $G(p^n)$, by the product of e and d in the following way.*

(4.1) *In the case D of order* 2, $S=G(p^n)\cup\{d\}$, $n\neq 0$,

$$ed\in G(p^{n-1})-G(p^{n-2}).^4$$

(4.2) *In the case D of order* 3, *D is of type* (2.1) *and* $S=G(p^n)\cup\{d_1, d_2\}$, $n\neq 0$.

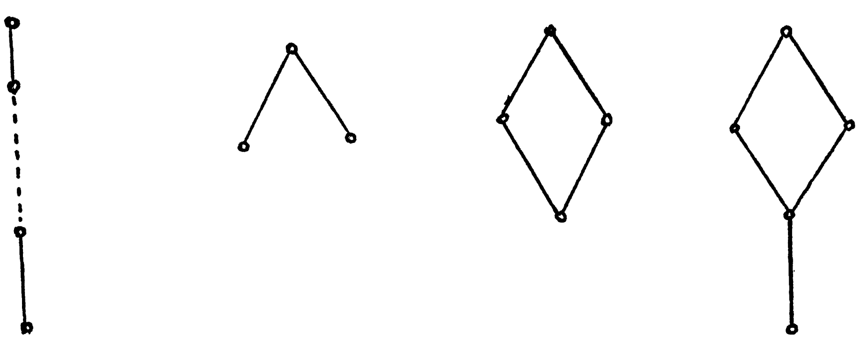
(4.2.1) $ed_1=ed_2\in G(p^n)-G(p^{n-1})$,

(4.2.2) $p^n\neq 2$, $ed_1\neq ed_2$; $ed_1, ed_2\in G(p^n)-G(p^{n-1})$.

(4.3) *In the case D of order* 4, $S=G(p^n)\cup\{d_1, d_2, d_3\}$, $d_2^2=d_3^2=d_1$, $n\neq 0$, $p\neq 2$.

$$\left.\begin{array}{l}(4.3.1)\ D \text{ of } (2.2.1)\\(4.3.2)\ D \text{ of } (2.2.2)\\(4.3.3)\ D \textit{ of } (2.2.3)\end{array}\right\}\ ed_2=ed_3\in G(p^n)-G(p^{n-1}).$$

According to the above-mentioned theorems, we see that if S is a finite Γ^*-semigroup, the finite C_0-semilattice $L(S)$ satisfies Jordan-Dedekind condition (or J-condition cf. [8]). Generally a finite C_0-semilattice K satisfying J-condition is called a C_0J-semilattice. Let δ denote the dimension of K (cf. [8]), λ the breadth, i.e., the number of the maximal chains in K, and μ the order, i.e., the number of elements of K.



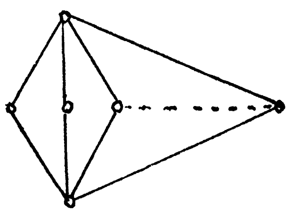
Γ-Semigroups | Idempotent Semi-groups of order 2 | (2.1), (3.1) | (2.2)

THEOREM 4. *A finite C_0J-semilattice K is isomorphic to certain $L(S)$ for some finite Γ^*-semigroup S if and only if δ, λ, and μ satisfy the following conditions.*

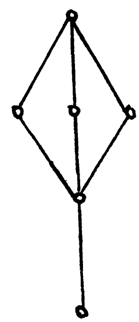
(5.1) $\delta+\lambda-\mu=0$,

(5.2) $\lambda=\alpha+1$ *where* $\alpha=0$ *or* 1 *or any prime number*,

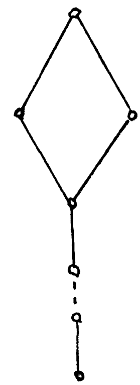
[4] $A-B$ denotes the set of elements of A which are not in B.



(3.1), (3.2)



(3.3)



(4.1), (4.2), (4.3)

$$(5.3)\quad \begin{cases} \textit{if } \lambda = 1 \textit{ or } 2, \textit{ then } \delta \textit{ can be taken as an arbitrary positive integer,} \\ \textit{if } \lambda = 3, \textit{ then } \delta = 2 \textit{ or } 3, \\ \textit{if } \lambda = p+1, \ p \textit{ being a prime number } > 2, \textit{ then } \delta = 2. \end{cases}$$

Finally we shall show the diagrams of $L(S)$ for a finite Γ^*-semigroup S.

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