NOTE ON Γ*-SEMIGROUPS

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The system \( L(S) \) of all nonvoid subsemigroups of a semigroup \( S \) is generally a semilattice\(^1\) with respect to the inclusion relation. \( L(S) \) is called the subsemigroup semilattice of \( S \). In the previous paper \([1]\) we determined all the \( \Gamma \)-semigroups,\(^2\) i.e., the semigroups whose subsemigroup semilattices are chains. In detail, all the types of \( \Gamma \)-semigroups are

\[
\begin{align*}
(1.1) & \text{ cyclic groups } G(p^n) \text{ of order of prime power,} \\
(1.2) & \text{ quasi-cyclic groups } G(p^n), \\
(1.3) & \text{ unipotent semigroups generated by } d \text{ with each of the following defining relations: } \\
& \quad (1.3.1) \ d^2 = d^3, \\
& \quad (1.3.2) \ d^3 = d^4, \\
& \quad (1.3.3) \ d^2 = d^{p^{m+2}}, \ p \text{ prime,} \\
& \quad (1.3.4) \ d^3 = d^{p^{m+3}}, \ p \text{ prime } \neq 2.
\end{align*}
\]

In the present note, we shall define \( \Gamma^* \)-semigroups as generalizations of \( \Gamma \)-semigroups and shall report the structure of \( \Gamma^* \)-semigroups except for a part of infinite \( \Gamma^* \)-groups. The proof will be omitted here but will be given elsewhere.\(^3\)

**DEFINITION.** A semigroup \( S \) is called a \( \Gamma^* \)-semigroup if every subsemigroup different from \( S \) is a \( \Gamma \)-semigroup.

\( S \) is a \( \Gamma^* \)-semigroup if and only if \( L(S) \) is a semilattice satisfying:

Any subset which contains the greatest element is a subsemilattice.

A semilattice of this kind is called a \( C_0 \)-semilattice. Obviously all the semigroups of order 2 are \( \Gamma^* \)-semigroups, and a homomorphic image of a \( \Gamma^* \)-semigroup is also a \( \Gamma^* \)-semigroup.

**LEMMA 1.** Every element of a \( \Gamma^* \)-semigroup is of finite order, that is, for any element \( x \) there is an idempotent \( e \) and a positive integer \( n \) such that \( x^n = e \).

**LEMMA 2.** A \( \Gamma^* \)-semigroup of order \( >2 \) is unipotent. (i.e., an idempotent element is unique).

Generally a unipotent semigroup any element of which is of finite

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\(^1\) By a semilattice we mean a partially ordered set in which there is a join of two elements.

\(^2\) In \([1]\) we called them \( \Gamma \)-monoids.

\(^3\) *Semigroups and their subsemigroups semilattices*, to appear.
order is determined by a group and a \( \Gamma \)-semigroup (i.e., a unipotent semigroup with zero) [2; 3]. By Lemmas 1 and 2, we can make the discussion proceed to \( \Gamma^* \)-\( Z \)-semigroups, \( \Gamma^* \)-groups, and then to the general cases.

**Theorem 1.** Any \( \Gamma^* \)-\( Z \)-semigroup is of order \( \leq 4 \). All the types of \( \Gamma^* \)-\( Z \)-semigroups other than \( \Gamma \)-semigroups are listed as follows:

1. {0, \( a \), \( b \)} of order 3 where \( xy = 0 \) for all \( x, y \),
2. {0, \( a \), \( b \), \( c \)} of order 4 defined as
   - (2.2.1) \( b^2 = c^2 = a \) and other products = 0.
   - (2.2.2) \( b^2 = cb = c^2 = a \) and other products = 0.
   - (2.2.3) \( b^2 = c^2 = bc = cb = a \), and other products = 0.

As far as the \( \Gamma^* \)-groups are concerned, we shall limit ourselves to the case of \( \Gamma^* \)-groups which are properly homomorphic to \( \Gamma \)-groups.

We can prove that any \( \Gamma^* \)-group which is properly homomorphic to a \( \Gamma \)-group has a normal subgroup of index of a prime number. Making use of the theory of finite groups [4; 5; 6], we have

**Theorem 2.** Any \( \Gamma^* \)-group, which is not a \( \Gamma \)-group and is homomorphic to a \( \Gamma \)-group of order \( > 1 \), has one of the following types.

1. The groups of order \( pq \) where \( p \) and \( q \) are different primes.
2. The elementary abelian group: \( G(p) \times G(p) \).
3. The generalized quaternion group of order 8.

Incidentally a finite \( \Gamma^* \)-group, which is not a \( \Gamma \)-group, is homomorphic to a \( \Gamma \)-group; a commutative \( \Gamma^* \)-group which is not a \( \Gamma \)-group is the direct product of two groups of prime order. Consequently we see that the result of Theorem 2 includes the cases where a \( \Gamma^* \)-group is homomorphic to a nontrivial finite group or a commutative group. However the problem of determination of the remaining case is still open.

Next, let \( S \) be a unipotent \( \Gamma^* \)-semigroup which is neither a group nor a \( Z \)-semigroup. Then we can prove that \( S \) must be finite. The kernel (i.e., the least ideal) of \( S \) is of type \( G(p^n) \), and the difference semigroup \( D \) of \( S \) modulo \( G(p^n) \), due to Rees [7] is a \( Z \)-semigroup which has one of the types (1.3.1), (2.1), (2.2.1), (2.2.2), (2.2.3).

Let \( e \) be the unique idempotent of \( S \), and let \( d \) be a generator of \( D \). \( G(p^{n-1}) \) will denote the subgroup of order \( p^{n-1} \) of \( G(p^n) \).

**Theorem 3.** When \( G(p^n) \) is given, we can determine all the unipotent \( \Gamma^* \)-semigroups, non \( \Gamma \)-semigroups, whose kernel is \( G(p^n) \), by the product of \( e \) and \( d \) in the following way.
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(4.1) In the case D of order 2, \( S = G(p^n) \cup \{d\}, n \neq 0, \)
\[ ed \in G(p^{n-1}) - G(p^{n-2}). \]

(4.2) In the case D of order 3, D is of type (2.1) and \( S = G(p^n) \)
\[ \cup \{d_1, d_2\}, n \neq 0. \]

(4.2.1) \( ed_1 = ed_3 \in G(p^n) - G(p^{n-1}), \)
(4.2.2) \( p^n \neq 2, ed_1 \neq ed_3; ed_1, ed_3 \in G(p^n) - G(p^{n-1}). \)

(4.3) In the case D of order 4, \( S = G(p^n) \cup \{d_1, d_2, d_3\}, d_2 = d_3 = d_1, \)
\[ n \neq 0, p \neq 2. \]

(4.3.1) D of (2.2.1) \[ ed_2 = ed_3 \in G(p^n) - G(p^{n-1}). \]
(4.3.2) D of (2.2.2) \[ D of (2.2.3) \]

According to the above-mentioned theorems, we see that if \( S \) is a
finite Γ*-semigroup, the finite \( C_0 \)-semilattice \( L(S) \) satisfies Jordan-
Dedekind condition (or J-condition cf. [8]). Generally a finite \( C_0 \)-
semilattice \( K \) satisfying J-condition is called a \( C_0J \)-semilattice. Let \( \delta \) denote the dimension of \( K \) (cf. [8]), \( \lambda \) the breadth, i.e., the number
of the maximal chains in \( K \), and \( \mu \) the order, i.e., the number of ele­
ments of \( K \).

\[ \begin{array}{ll}
\text{Γ-Semigroups} & \text{Idempotent Semi-} \\
(2.1), (3.1) & \text{groups of order 2} \\
(2.2) & \\
\end{array} \]

**Theorem 4.** A finite \( C_0J \)-semilattice \( K \) is isomorphic to certain \( L(S) \)
for some finite Γ*-semigroup \( S \) if and only if \( \delta, \lambda, \) and \( \mu \) satisfy the fol­
lowing conditions.

(5.1) \( \delta + \lambda - \mu = 0, \)
(5.2) \( \lambda = \alpha + 1 \) where \( \alpha = 0 \) or 1 or any prime number,

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\[ A - B \] denotes the set of elements of \( A \) which are not in \( B. \]
if \( \lambda = 1 \) or 2, then \( \delta \) can be taken as an arbitrary positive integer,

if \( \lambda = 3 \), then \( \delta = 2 \) or 3,

if \( \lambda = p + 1 \), \( p \) being a prime number > 2, then \( \delta = 2 \).

Finally we shall show the diagrams of \( L(S) \) for a finite \( \Gamma^* \)-semigroup \( S \).

References