ON THE MAXIMUM OF A NORMAL STATIONARY STOCHASTIC PROCESS[1]

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1. Let $x(t)$ with $-\infty < t < +\infty$ be the variables of a real, separable, normal and stationary stochastic process, such that $E[x(t)]=0$ and $E[x^2(t)]=1$. Let the covariance function of the process be

$$r(t) = E[x(t)x(0)] = \int_0^\infty \cos \lambda t f(\lambda) d\lambda,$$

and assume that the spectral density $f(\lambda)$ is of bounded variation in $(-\infty, \infty)$ and satisfies the condition

$$\int_0^\infty \lambda^2 (\log(1+\lambda))^a f(\lambda) d\lambda < \infty$$

for some $a>1$.

Then it is known (Hunt [5], Belayev [1]) that the sample functions $x(t)$ will almost certainly be everywhere continuous and have continuous first derivatives $x'(t)$. Consequently for every fixed $t>0$ the maximum

$$\max_{0 \leqq u \leqq t} x(u)$$

will be a random variable defined but for equivalence.

For the sake of typographical convenience, we write in the sequel simply $\max x(u)$, omitting the subscript $0 \leqq u \leqq t$, and similarly in respect of $\min x(u)$.

The object of this note is to prove the relation

$$(1) \qquad \lim_{t \to \infty} P\left[\left| \max x(u) - (2 \log t)^{1/2} \right| < \frac{\log \log t}{(\log t)^{1/2}} \right] = 1.$$

The notation $P[\cdots]$ denotes here, as throughout the sequel, the probability of the relation between the brackets.

A similar relation was recently given for the case of a normal stationary sequence x_n with $n=0, \pm 1, \cdots$ by Berman [2].

2. We shall first prove that

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$$(2) \qquad P\left[\max x(u) \leqq (2 \log t)^{1/2} - \frac{\log \log t}{(\log t)^{1/2}}\right] \to 0$$

as $t \to \infty$.

Let $c > 0$ be given, and define a random variable $y(u)$ by writing for any real u

$$y(u) = \begin{cases} 1 & \text{if } x(u) > c, \\ 0 & \text{if } x(u) \leqq c. \end{cases}$$

Then $y(u)$ will define a stationary process such that

$$E[y(u)] = P[x(u) > c] = \int_c^\infty \phi(x)dx,$$

$$E[y(u)y(v)] = P[x(u) > c,\ x(v) > c] = \int_c^\infty \int_c^\infty \phi(x, y; r)dxdy,$$

where

$$\phi(x) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{x^2}{2}\right),$$

$$\phi(x, y; r) = \frac{1}{2\pi(1 - r^2)^{1/2}} \exp\left(-\frac{x^2 - 2rxy + y^2}{2(1 - r^2)}\right),$$

$$r = r(u - v).$$

It follows (cf. e.g. Loève [6, pp. 472, 520]) that the integral

$$z(t) = \int_0^t y(u)du$$

is defined both in quadratic mean and as a sample function integral, and that the two integrals coincide, but for equivalence. Then $z(t)$ will, with probability 1, be equal to the Lebesgue measure of the set of points u in $[0, t]$ such that $x(u) > c$. Thus $z(t) \geqq 0$ with probability 1, and

$$(3) \qquad P[z(t) = 0] = P[\max x(u) \leqq c].$$

For all sufficiently large c we have (Loève, l.c.)

$$(4) \qquad E[z(t)] = t\int_c^\infty \phi(x)dx > \frac{t}{3c} \exp\left(-\frac{c^2}{2}\right),$$

and further

$$E[z^2(t)] = \int_0^t \int_0^t dudv \int_c^\infty \int_c^\infty \phi(x, y; r)dxdy$$

with $r = r(u-v)$.

For any fixed r in $(-1, 1)$ we have the identity

$$\int_c^\infty \int_c^\infty \phi(x, y; r)dxdy$$

$$= \left(\int_c^\infty \phi(x)dx\right)^2 + \frac{1}{2\pi}\int_0^r \exp\left(-\frac{c^2}{1+w}\right)\frac{dw}{(1-w^2)^{1/2}}.$$

(For $r=0$ the identity is obvious, and some calculation will show that the derivatives of both sides with respect to r are equal.)

It then follows that the variance of $z(t)$ is

$$\begin{aligned} \operatorname{Var}[z(t)] &= \frac{1}{2\pi}\int_0^t \int_0^t dudv \int_0^{r(u-v)} \exp\left(-\frac{c^2}{1+w}\right)\frac{dw}{(1-w^2)^{1/2}} \\ &< \frac{1}{\pi^2}\int_0^t \int_0^t |r(u-v)| \exp\left(-\frac{c^2}{1+|r(u-v)|}\right)dudv. \end{aligned} \tag{5}$$

From our assumptions concerning the spectral density $f(\lambda)$, it follows that there exist positive constants k and m such that

$$|r(t)| < \frac{k}{|t|} \qquad \text{for all } t,$$

$$|r(t)| \leqq 1 - m^2t^2 \quad \text{for } |t| \leqq 2k.$$

(The latter inequality is easily proved by means of Cramér [4, Lemma 1].)

Dividing the domain of integration in (5) into two parts, defined respectively by $|u-v| > 2k$ and $|u-v| \leqq 2k$, and using in each part the appropriate inequality for $|r(u-v)|$, we obtain from (5) by some straightforward estimation

$$\operatorname{Var}[z(t)] < 2kt \log t \exp\left(-\frac{2c^2}{3}\right) + \frac{2\pi^{1/2}}{m}\cdot\frac{t}{c}\exp\left(-\frac{c^2}{2}\right). \tag{6}$$

Now the Tchebychev inequality gives

$$P[z(t) = 0] \leqq \frac{\operatorname{Var}[z(t)]}{E^2[z(t)]}.$$

Taking

$$c = (2 \log t)^{1/2} - \frac{\log \log t}{(\log t)^{1/2}},$$

we then obtain from (3), (4) and (6)

$$P[\max x(u) \leqq c] < A((\log t)^2 t^{-1/8} + (\log t)^{1/2 - 2^{1/2}})$$

where A is independent of t. Since the second member obviously tends to zero as $t \to \infty$, (2) is proved.

3. It now remains to prove that

$$(7) \qquad P\left[\max x(u) \geqq (2 \log t)^{1/2} + \frac{\log \log t}{(\log t)^{1/2}}\right] \to 0$$

as $t \to \infty$. For any $c > 0$ we evidently have

$$\begin{aligned} P[\max x(u) \geqq c] &= P[\min x(u) \leqq c \leqq \max x(u)] + P[\min x(u) > c] \\ &= P_1 + P_2. \end{aligned}$$

P_1 is, for any continuous sample function $x(u)$, the probability of at least one "crossing" with the level c within $[0, t]$, i.e., the probability that there is at least one point u in $[0, t]$ such that $x(u) = c$. Let N denote the total number of such points, and write $p_n = P[N = n]$ for $n = 0, 1, \cdots$. Then

$$(8) \qquad P_1 = p_1 + p_2 + \cdots \leqq p_1 + 2p_2 + \cdots = E[N].$$

However, it is known (Bulinskaya [3]) that under the present conditions

$$(9) \qquad E[N] = \frac{(\lambda_2)^{1/2}}{\pi} t \exp\left(-\frac{c^2}{2}\right),$$

where λ_2 denotes the second moment of $f(\lambda)$. Further

$$(10) \qquad \begin{aligned} P_2 &= P[\min x(u) > c] \leqq P[x(0) > c] \\ &= \int_c^\infty \phi(x) dx < \frac{1}{c(2\pi)^{1/2}} \exp\left(-\frac{c^2}{2}\right). \end{aligned}$$

Taking now

$$c = (2 \log t)^{1/2} + \frac{\log \log t}{(\log t)^{1/2}},$$

it follows from (8), (9) and (10) that P_1 and P_2 both tend to zero as $t \to \infty$, so that (7) is proved. Finally, the result (1) follows from (2) and (7).

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ON THE MAXIMUM TRANSFORM AND SEMIGROUPS OF TRANSFORMATIONS

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1. **Introduction.** The problem of determining the maximum of the function

$$F(x_1, x_2, \cdots, x_N) = \sum_{i=1}^{N} g_i(x_i) \tag{1.1}$$

over the domain D_N defined by $\sum_{i=1}^{N} x_i = x$, $x_i \geqq 0$, is one with various ramifications and applications. Analytic solutions and computational algorithms have been obtained in a number of ways; see Karush [**7**], Bellman [**2**], Bellman and Karush [**3**]. Let us now discuss a new way of generating solutions of (1.1). Let $g(x, a)$ be a scalar function of the scalar variable x and the M-dimensional vector a with the group property that

$$\max_{x_1+x_2=x} [g(x_1, a) + g(x_2, b)] = g(x, h(a, b)) \qquad (x_1, x_2 \geqq 0), \tag{1.2}$$

where $h(a, b)$ is a known function of a and b. It follows inductively that

$$\max_{D_N} \left[\sum_{k=1}^{N} g(x_k, a^{(k)}) \right] = g(x, h(a^{(1)}, a^{(2)}, \cdots, a^{(N)})), \tag{1.3}$$

where D_N is as above, and $h(a^{(1)}, a^{(2)}, \cdots, a^{(N)})$ is obtained from