

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HYPERBOLIC INEQUALITIES¹

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We consider the asymptotic behavior of solutions of inequalities of the form

$$(1.1) \quad |Lu|^2 \leq c_1 |u|^2 + c_2 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 + c_3 \left| \frac{\partial u}{\partial t} \right|^2$$

where

$$(1.2) \quad L = A - \frac{\partial^2}{\partial t^2} + b$$

and A is a second order elliptic operator. The asymptotic behavior of solutions of parabolic inequalities and related problems have been considered by Agmon and Nirenberg [1], Cohen and Lees [2], Lax [3], and the author [4].

Let D be a bounded domain in E^n and suppose $u(x_1, \dots, x_n, t) = u(x, t)$ is a solution of (1.1) in the cylindrical region $R = D \times I$ where I is the half-infinite interval $0 \leq t < \infty$. We shall study the behavior as $t \rightarrow \infty$ in R of those solutions u which satisfy the additional condition

$$(1.3) \quad u = 0 \quad \text{on} \quad \Gamma \times I$$

where Γ is the boundary of D .

We introduce the notation

$$\begin{aligned} (u, v) &= \int_R u(x, t)v(x, t) dx dt, \\ \|u\| &= (u, u)^{1/2}, \\ \|u\|_1^2 &= \int_R \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 dx dt, \\ \|u\|_{D,1}^2 &= \int_D \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 dx. \end{aligned}$$

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The elliptic operator A has the form

$$A = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right), \quad a_{ij} = a_{ji},$$

where the $a_{ij} = a_{ij}(x, t)$ are C^1 functions of x and t .

A function $v(x, t)$ defined in R is said to satisfy *Conditions B* if

$$(1.4) \quad \begin{aligned} v &= 0 \quad \text{on} \quad \Gamma \times I, \\ \lim_{t \rightarrow \infty} t^\alpha \|v\|_{D,1} &= 0 \quad \text{for every } \alpha > 0. \end{aligned}$$

The operator L is said to satisfy *Conditions C* if

$$(1.5) \quad \frac{\partial}{\partial t} (a_{ij}) = O\left(\frac{1}{t}\right) \quad \text{for } i, j = 1, 2, \dots, n.$$

$$(1.6a) \quad \frac{\partial b}{\partial t} \leq 0 \quad \text{for all sufficiently large } t.$$

If (1.5) holds and (1.6a) is replaced by

$$(1.6b) \quad \frac{\partial b}{\partial t} = O(t^{-3}).$$

We say that *Conditions C'* are satisfied.

LEMMA 1. *If $v(x, t)$ satisfies Conditions B and the operator L satisfies Conditions C or C' then for all sufficiently large α we have*

$$\alpha^4 \|t^{\alpha-2} v\|^2 + \alpha^2 \|t^{\alpha-1} v\|_1^2 \leq m_0 \|t^\alpha Lv\|^2$$

where m_0 is a positive constant depending only on L .

LEMMA 2. *Under the hypotheses of Lemma 1 we have*

$$\alpha^{1/2} \|t^{\alpha-1} v_t\| \leq m_1 \|t^\alpha Lv\|$$

for all sufficiently large α ; m_1 is a positive constant depending only on L .

THEOREM 1. *Let $u(x, t)$ satisfy in R the differential inequality (1.1) and suppose Conditions B and Conditions C or C' hold. If in addition*

$$(1.7) \quad c_1(t) = O(t^{-2}), \quad c_2(t), c_3(t) = O(t^{-1})$$

then $u \equiv 0$ in R .

Theorem 1 follows from Lemmas 1 and 2 by standard arguments.

If we assume that the solution of (1.1) decays more rapidly than stated in *Conditions B* then the hypotheses on the coefficients of L

and on $c_i(t)$, $i = 1, 2, 3$ may be relaxed considerably.

A function $v(x, t)$ defined in R is said to satisfy *Conditions E* if

$$(1.8) \quad \begin{aligned} v &= 0 \quad \text{on } \Gamma \times I, \\ \lim_{t \rightarrow \infty} e^{\lambda t} \|v\|_{D,1} &= 0 \quad \text{for every } \lambda > 0. \end{aligned}$$

LEMMA 3. *Suppose v satisfies Conditions E and vanishes for $0 \leq t \leq \epsilon$ for some $\epsilon > 0$. If the coefficients of L have bounded first derivatives then for all sufficiently large $\lambda > 0$ we have*

$$\lambda^4 \|e^{\lambda t} v\|^2 + \lambda^2 \|e^{\lambda t} v\|_1^2 \leq m_2 \|e^{\lambda t} Lv\|^2$$

where m_2 is a positive constant depending only on L .

LEMMA 4. *Under the hypotheses of Lemma 3 we have*

$$\lambda^{1/2} \|e^{\lambda t} v_i\| \leq m_3 \|e^{\lambda t} Lv\|$$

where m_3 is a positive constant depending only on L .

THEOREM 2. *Let $u(x, t)$ satisfy in R the differential inequality (1.1) and suppose Conditions E hold. If the coefficients of L have bounded first derivatives and if $c_i(t)$, $i = 1, 2, 3$, are bounded then $u \equiv 0$ in R .*

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