ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HYPERBOLIC INEQUALITIES[1]

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We consider the asymptotic behavior of solutions of inequalities of the form

$$(1.1)\qquad |Lu|^2 \leqq c_1 |u|^2 + c_2 \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 + c_3 \left| \frac{\partial u}{\partial t} \right|^2$$

where

$$(1.2)\qquad L = A - \frac{\partial^2}{\partial t^2} + b$$

and A is a second order elliptic operator. The asymptotic behavior of solutions of parabolic inequalities and related problems have been considered by Agmon and Nirenberg [**1**], Cohen and Lees [**2**], Lax [**3**], and the author [**4**].

Let D be a bounded domain in E^n and suppose $u(x_1, \cdots, x_n, t) = u(x, t)$ is a solution of (1.1) in the cylindrical region $R = D \times I$ where I is the half-infinite interval $0 \leqq t < \infty$. We shall study the behavior as $t \to \infty$ in R of those solutions u which satisfy the additional condition

$$(1.3)\qquad u = 0 \quad \text{on} \quad \Gamma \times I$$

where Γ is the boundary of D.

We introduce the notation

$$(u, v) = \int_R u(x, t) v(x, t) dx dt,$$

$$\|u\| = (u, u)^{1/2},$$

$$\|u\|_1^2 = \int_R \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx dt,$$

$$\|u\|_{D,1}^2 = \int_D \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx.$$

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The elliptic operator A has the form

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right), \qquad a_{ij} = a_{ji},$$

where the $a_{ij}=a_{ij}(x, t)$ are C^1 functions of x and t.

A function $v(x, t)$ defined in R is said to satisfy *Conditions* B if

$$\text{(1.4)} \qquad \begin{gathered} v = 0 \quad \text{on} \quad \Gamma \times I, \\ \lim_{t\to\infty} t^{\alpha}\|v\|_{D,1} = 0 \qquad \text{for every } \alpha > 0. \end{gathered}$$

The operator L is said to satisfy *Conditions* C if

$$\text{(1.5)} \qquad \frac{\partial}{\partial t}(a_{ij}) = O\left(\frac{1}{t}\right) \qquad \text{for } i, j = 1, 2, \cdots, n.$$

$$\text{(1.6a)} \qquad \frac{\partial b}{\partial t} \leqq 0 \qquad \text{for all sufficiently large } t.$$

If (1.5) holds and (1.6a) is replaced by

$$\text{(1.6b)} \qquad \frac{\partial b}{\partial t} = O(t^{-3}).$$

We say that *Conditions* C′ are satisfied.

LEMMA 1. *If $v(x, t)$ satisfies Conditions* B *and the operator L satisfies Conditions* C *or* C′ *then for all sufficiently large α we have*

$$\alpha^4\|t^{\alpha-2}v\|^2 + \alpha^2\|t^{\alpha-1}v\|_1^2 \leqq m_0\|t^{\alpha}Lv\|^2$$

where m_0 is a positive constant depending only on L.

LEMMA 2. *Under the hypotheses of Lemma* 1 *we have*

$$\alpha^{1/2}\|t^{\alpha-1}v_t\| \leqq m_1\|t^{\alpha}Lv\|$$

for all sufficiently large α; m_1 is a positive constant depending only on L.

THEOREM 1. *Let $u(x, t)$ satisfy in R the differential inequality* (1.1) *and suppose Conditions* B *and Conditions* C *or* C′ *hold. If in addition*

$$\text{(1.7)} \qquad c_1(t) = O(t^{-2}), \quad c_2(t), c_3(t) = O(t^{-1})$$

then $u\equiv 0$ in R.

Theorem 1 follows from Lemmas 1 and 2 by standard arguments.

If we assume that the solution of (1.1) decays more rapidly than stated in *Conditions* B then the hypotheses on the coefficients of L

and on $c_i(t)$, $i=1, 2, 3$ may be relaxed considerably.

A function $v(x, t)$ defined in R is said to satisfy *Conditions* E if

$$(1.8)\qquad \begin{aligned} v &= 0 \quad \text{on} \quad \Gamma \times I, \\ \lim_{t\to\infty} e^{\lambda t}\|v\|_{D,1} &= 0 \qquad \text{for every } \lambda > 0. \end{aligned}$$

LEMMA 3. *Suppose v satisfies Conditions* E *and vanishes for* $0 \leqq t \leqq \epsilon$ *for some* $\epsilon > 0$. *If the coefficients of L have bounded first derivatives then for all sufficiently large* $\lambda > 0$ *we have*

$$\lambda^4\|e^{\lambda t}v\|^2 + \lambda^2\|e^{\lambda t}v\|_1^2 \leqq m_2\|e^{\lambda t}Lv\|^2$$

where m_2 *is a positive constant depending only on L.*

LEMMA 4. *Under the hypotheses of Lemma* 3 *we have*

$$\lambda^{1/2}\|e^{\lambda t}v_t\| \leqq m_3\|e^{\lambda t}Lv\|$$

where m_3 *is a positive constant depending only on L.*

THEOREM 2. *Let* $u(x, t)$ *satisfy in R the differential inequality* (1.1) *and suppose Conditions E hold. If the coefficients of L have bounded first derivatives and if* $c_i(t)$, $i=1, 2, 3$, *are bounded then* $u\equiv 0$ *in R.*

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