

# SOLUTIONS OF EQUATIONS OVER GROUPS

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A polynomial equation of degree  $n$  over a field  $K$  can always be solved in a suitable extension  $K'$  of degree at most  $n$  over  $K$ . One might expect to find a corresponding result for groups; it is the purpose of this note to show that this is, in fact, the case.

The analogous problem for groups is to find a *solution* for the equation

$$(1) \quad x^{n_1}g_1x^{n_2}g_2 \cdots x^{n_k}g_k = 1$$

in the *unknown*  $x$ , where the  $g_i$  are elements of a group  $G$  and all the  $n_i$  are non-negative, i.e., to find an element  $g'$  in a group  $G'$  in which  $G$  is embedded such that

$$g'^{n_1}g_1g'^{n_2}g_2 \cdots g'^{n_k}g_k = 1$$

in  $G'$ . It will be shown that (1) can always be solved over any group  $G$ , with a solution in  $G'$ , where  $G'$  is, in some sense, an extension group of "degree  $n$ " over  $G$ , for  $n = \sum_{i=1}^k n_i$ . This result (coupled with the fact that the equation

$$x^{-1}g_1xg_2 = 1$$

is solvable over any torsion-free group [2]) gives hope for the conjecture that any equation is solvable over a torsion-free group.

The solution we will give for (1) uses the construction used by Baumslag [1] to solve the equation  $x^n g = 1$  (cf. [3]).

Without loss of generality, we may restrict our attention to equations of the form

$$(2) \quad xa_0xa_1 \cdots xa_{n-1} = 1, \quad a_i \in G,$$

where some of the  $a_i$  are possibly 1. The solution of (2) will be constructed in the wreath product of  $G$  and a cyclic group  $C$  of order  $n$ . To fix notation, we will outline the definition of the wreath product,  $\text{GWr}C$  (cf. [2]): Let  $G^C$  be the group of all mappings  $\{f\}$  of  $C$  into  $G$  with  $ff' \in G^C$  defined by  $ff'(t) = f(t)f'(t)$ , for all  $t \in C$ .  $\text{GWr}C$  is the group composed of the set  $\{sf \mid s \in C, f \in G^C\}$  with

$$sf \cdot s'f' = ss'f^s f',$$

where  $f^s(t) = f(ts^{-1})$  for all  $t \in C$ . We will embed  $G$  in  $\text{GWr}C$  by identifying  $1'g^o \in \text{GWr}C$  with  $g \in G$ , where  $g^o(t) = g$  for all  $t \in C$  and  $1'$  is the neutral element of  $C$ .

The main result of this note is stated as follows:

**THEOREM 1.** *Let  $G$  be an arbitrary group,  $C = gp(c)$  be a cyclic group of order  $n$ . A solution of equation (2) is given by  $c^{-1}f \in \text{GWrC}$ , where  $f(c^i) = a_i^{-1}$ ,  $i = 0, 1, \dots, n-1$ .*

In other words, Theorem 1 states that

$$(3) \quad (c^{-1}f) \cdot (1'a_0^\circ) \cdot (c^{-1}f) \cdot (1'a_1^\circ) \cdot \dots \cdot (c^{-1}f) \cdot (1'a_{n-1}^\circ) = 1'1^\circ,$$

(1 is the neutral element of  $G$ ), which can be verified by a straightforward application of the definition of  $\text{GWrC}$ .

There are several properties of a group  $G$  which are inherited by  $\text{GWrC}$ . For instance: if  $G$  is finite,  $\text{GWrC}$  is finite; if  $G$  has finite exponent  $m$ ,  $\text{GWrC}$  has exponent  $mn$ ; if  $G$  is soluble,  $\text{GWrC}$  is soluble of length at most one greater than  $G$ .

A group which has the property that every equation of type (2) has a solution in the group itself will be called a  *$P$ -algebraically closed group*. Such a group is also divisible, i.e., contains a solution for every equation  $x^n g = 1$ . Some of the results in [1] can be extended immediately to  $P$ -algebraically closed groups. In particular, Corollary 4.3 and Theorem 4.4 of [1], respectively, have the following extensions:

**COROLLARY 1** (*cf.*, also, Neumann [3]). *Every group  $G$  can be embedded in a  $P$ -algebraically closed group.*

**THEOREM 2.** *Every periodic group can be embedded in a periodic  $P$ -algebraically closed group.*

For the solution of (2) it is sufficient, of course, to consider the subgroup  $W'$  of  $\text{GWrC}$  generated by  $G$  and  $c^{-1}f$  rather than the whole of  $\text{GWrC}$ .  $W'$  has, in fact, the same solubility length as  $G$ ; this result can be checked rather easily here. Consequently, we can state

**THEOREM 3.** *If  $G$  is soluble of length  $q$ , a solution of equation (2) can be found in an overgroup  $W'$  of  $G$  which is soluble of length  $q$ .*

#### REFERENCES

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3. B. H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. **18** (1943), 12-20.

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