A DUALITY THEORY FOR CONVEX PROGRAMS
WITH CONVEX CONSTRAINTS

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The existence of a solution to the problem of minimizing a convex function subject to restriction of the variables to a closed convex set in \( n \)-space ("convex programming") has been characterized (for suitable differentiability conditions) by the Kuhn-Tucker theorem [5]. In general, no dual programming problem (not involving the variables of the direct problem) has been associated with this situation except in the linear programming case, and very recently by E. Eisenberg in [3], for homogeneity of order one in the function and linear inequality constraints, and by R. J. Duffin [2] in an inverse manner for a highly specialized problem.

Starting with a little known paper of A. Haar [4] in the light of current linear programming constructs (e.g., "regularization" [1]), we effect a generalization of these ideas (with maximal finite algebra and minimal topology) so that a dual theory practically as straightforward as linear programming theory is obtained, and which includes a dual theorem covering the most general convex programming situation (e.g. no differentiability conditions qualifying the convex function or constraints, or homogeneity, etc.).

This general theorem is made possible by associating a suitably restricted, usually infinite-dimensional space problem with the minimization problem in \( n \)-space instead of the usual association of another finite \( m \)-space problem. The space we use is a "generalized finite sequence space" (g.f.s.s.), defined with respect to an index set \( I \) of arbitrary cardinality as the vector space, \( S \), of all vectors \( \lambda = [\lambda_i: i \in I] \) over an ordered field \( F \) with only finitely many nonzero entries.

Such spaces possess the following key characteristics for linear programming of ordinary \( n \)-spaces. Let \( V \) be a vector space over \( F \) and consider a collection of vectors: \( P_0, P_1; i \in I \) in \( V \). Let \( R \) be the subspace spanned by these vectors, and let

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\[ \Lambda = \left\{ \lambda \in S : \sum_{i \in I} \lambda_i P_i = P_0, \lambda \geq 0 \right\}. \]

Clearly \( \Lambda \) is convex in \( S \) and we have (assuming \( V \) finite-dimensional):

**Theorem 1.** \( \lambda \neq 0 \) is an extreme point of \( \Lambda \) in \( S \) if and only if the nonzero coordinates of \( \lambda \) correspond to coefficients of linearly independent vectors in \( R \).

**Theorem 2.** \( \Lambda \) is generated by its extreme points if and only if for any \( x \in S, x \neq 0, \sum_{i \in I} \alpha_i P_i = 0 \) implies some \( \alpha_r \) and some \( \alpha_s \) are of opposite signs.

**Remark.** \( \Lambda \) need not be bounded as in \( n \)-space. ("Bounded" means there exists \( M \in F \) such that \( \sum_{i} |\lambda_i| \leq M \) for all \( \lambda \) in the set.)

These theorems can be proved in similar fashion to their finite space forms due respectively to Charnes and to Charnes-Cooper (see [1]).

By "dual semi-infinite programs" we mean the following pair of problems formed from the same data:

\[
\begin{align*}
\text{I} & \quad \min u^T P_0 \\
\text{subject to} & \quad u^T P_i \geq c_i, \; i \in I
\end{align*}
\]

\[
\begin{align*}
\text{II} & \quad \max \sum_{i \in I} c_i \lambda_i \\
\text{subject to} & \quad \sum_{i \in I} P_i \lambda_i = P_0 \\
& \quad \lambda \in S, \lambda \geq 0.
\end{align*}
\]

We restrict ourselves now to the real field and to semi-infinite programs whose \( \{P_i, c_i\} \) are "canonically closed" in the sense that in an equivalent inequality system in which the new \( \{P_i, c_i\} \) form a bounded set, e.g. by dividing each inequality by some \( d_i > 0 \), the set is also closed. We call such programs "dual Haar programs."

We require next the inhomogeneous (inequality system) theorem of Haar [4].

**Theorem 3.** Let \( u^T P_i \geq c_i, \; i \in I \) be a canonically closed system. If \( u^T P \geq c \) holds whenever \( u^T P_i \geq c_i \) for all \( i \in I \), then there exist \( \lambda_k \geq 0, \lambda_0 \geq 0 \), with at most \( n+1 \) nonzero such that

\[ u^T P - c = \sum_{k} \lambda_k (u^T P_k - c_k) + \lambda_0. \]

Haar does not specifically use the notion of canonical closure, but as counter-examples show he must have intended something of this sort. By use of Theorem 3 we obtain the following lemma.
LEMMA 1. For Haar programs if both I and II are consistent, then
$$\inf u^TP_0 = \sup \sum_{i \in I} u^TP_i \lambda_i = \sum_{i \in I} c_i \lambda_1^*$$
for some $\lambda^* \in \Lambda$.

Hence we conclude

THEOREM 4 (EXTENDED DUAL THEOREM). For any pair of dual Haar programs precisely one of the following occurs.

(i) $\sup \sum_{i \in I} c_i \lambda_i = \infty$ and I is inconsistent.
(ii) $\inf u^TP_0 = -\infty$ and II is inconsistent.
(iii) I and II are both inconsistent.
(iv) $\inf u^TP_0 = \sup \sum_{i \in I} c_i \lambda_i^*$ for some $\lambda_i^* \in \Lambda$.

REMARK. Only the Farkas-Minkowski property of Theorem 3 is employed to obtain Theorem 4. Canonical closure is a sufficient but not a necessary condition for this.

To obtain the general convex programming dual theorem, we move the functional into the constraints and replace it with a linear function as follows. Suppose the direct problem is: $\min C(u)$ subject to $G(u) \geq 0$, where $G = (\cdots, G_i(u), \cdots)$ is a finite vector of concave functions which defines the convex set $W$ of the $u$'s. Let $u^TP_i \geq c_i$, $i \in I$ be a system of supports for $W$, and $z - u^TQ_a \geq d_a$, $a \in A$ be a system of supports for $z - C(u) \geq 0$. Then the direct problem may be rewritten as:

$$\min z, \quad \text{subject to} \quad z - u^TQ_a \geq d_a, \quad u^TP_i \geq c_i, \quad a \in A, \; i \in I.$$ 

Thus we have

THEOREM 5. Assuming the Farkas-Minkowski property for this system, the extended dual theorem applies to the following dual programs:

$$\begin{align*}
\min z & \quad \max \sum_a d_a \eta_a + \sum_i c_i \lambda_i \\
\text{subject to} \quad z - u^TQ_a & \geq d_a \\
\quad & \quad \sum_a \eta_a = 1 \\
\quad & \quad u^TP_i \geq c_i \\
\quad & \quad - \sum_a Q_a \eta_a + \sum_i P_i \lambda_i = 0 \\
\quad & \quad \eta_a, \lambda_i \geq 0.
\end{align*}$$

Complete generality may now be obtained since an arbitrary semi-infinite program may be replaced by a Haar program according to the following observation:

THEOREM 6. The canonical closure $u^TP_i \geq c_i$, $i \in \mathcal{I}$ of the system $u^TP_i \geq c_i$, $i \in I$, has precisely the same set of solutions $\{u\}$, where $\mathcal{I} \supseteq I$. 

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denotes the increased index set to index limit points of the \((P_i, c_i)\) not indexed by \(I\).

REFERENCES