1. Introduction. It is known that if $V$ is an isometry on a (complex) Hilbert space $\mathcal{H}$ onto a subspace $R$ of $\mathcal{H}$, then

$$\mathcal{H} = \sum_{k=0}^{\infty} V^k(R^\perp) + \bigcap_{k=0}^{\infty} V^k(\mathcal{H}),$$

where the two subspaces on the right-hand side are orthogonal, and $R^\perp$ is "wandering for $V$," i.e. $V^j(R^\perp) \perp V^k(R^\perp), j \neq k$. The identity (1.1) closely resembles the Wold decomposition of the "present and past subspace" of a weakly stationary stochastic process into its "innovation subspaces" and the "remote past" cf. [10, 6.10]. Interpreting $k$ as the time, we shall therefore speak of (1.1) as the Wold decomposition of $\mathcal{H}$ due to $V$ or (equivalently) due to the discrete semi-group $(V^k, k \geq 0)$, and refer to $V^k(R^\perp), k \geq 0$, as the innovation subspaces of $\mathcal{H}$, and to $\bigcap_{k=0}^{\infty} V^k(\mathcal{H})$ as the remote subspace of $\mathcal{H}$ engendered by the semi-group.

In this note our purpose is to obtain the analogous decomposition of $\mathcal{H}$ due to a strongly continuous semi-group $(S_t, t \geq 0)$ of isometries on $\mathcal{H}$ (6.5 below). We shall derive this by applying (1.1) to the Cayley transform $V$ of $H$, where $iH$ is the infinitesimal generator of the semi-group, and then replacing the direct sum $\bigcap_{k=0}^{\infty} V^k(\mathcal{H})$ of innovation subspaces, occurring in (1.1), by a direct integral of "differential innovation subspaces."

2. The associated discrete semi-group. Let $(S_t, t \geq 0)$ be a strongly continuous semi-group of isometries on $\mathcal{H}$ into $\mathcal{H}$, and let $iH$ be its infinitesimal generator. Then

$$S'_t = S_t iH = iHS_t, \quad \text{on } \mathcal{D}, \quad t \geq 0,$$

where $\mathcal{D}$, the domain of $H$, is a linear manifold everywhere dense in $\mathcal{H}$. From the work of J. L. B. Cooper [1], (cf. also [5]) we know that

(a) $H$ is maximal symmetric with deficiency index $(0, \alpha)$,
(b) \( H + iI \) is one-one on \( \mathcal{D} \) onto \( \mathcal{X} \),

\[
(2.2) \quad (H+iI)^{-1} = \frac{1}{i} \int_{0}^{\infty} e^{-iS} dt
\]

(c) \( (H+iI)^{-1} \) is one-one and bounded on \( \mathcal{X} \) onto \( \mathcal{D} \) and \( |(H+iI)^{-1}| \leq 1 \),

(d) \( H - iI \) is one-one on \( \mathcal{D} \) onto a (closed) subspace \( R \).

Now let \( V \) be the Cayley transform of \( H \):

\[
V = c(H) = (H - iI)(H + iI)^{-1}, \quad \text{on } \mathcal{X}.
\]

It follows from the work of von Neumann, cf. [9, Chapter IX], that

(a) \( V \) is an isometry on \( \mathcal{X} \) onto \( R \),

(b) \( I - V = 2i(H + iI)^{-1} = 2 \int_{0}^{\infty} e^{-iS} dt \),

\[
(2.3) \quad H = i(I + V)(I - V)^{-1} \quad \text{on } \mathcal{D},
\]

(c) \( S_t V^k = V^k S_t \quad \text{on } \mathcal{X}, \quad t \geq 0, k \geq 0. \)

We shall call \( (V^k, k \geq 0) \) the discrete semi-group of isometries associated with the given semi-group \( (S_t, t \geq 0) \). In the rest of §2 we shall formulate the basic relationship between the \( S_t \) and the \( V^k \).

The \( S_t \) are expressible in terms of \( H \) by the exponential formula, cf. [5],

\[
S_t = \lim_{n \to \infty} \exp(iHJ_n), \quad \text{strongly on } \mathcal{X},
\]

\[
(2.4) \quad \text{where } J_n = \left( I - \frac{1}{n} iH \right)^{-1}.
\]

Since \( J_n \) is a bounded operator, so therefore is \( iHJ_n = n(J_n - I) \). Hence \( \exp(iHJ_n) \) has a power series expansion, from which we get the following expression for \( S_t \) in terms of \( V^k \):

\[
S_t = e^{-tI} + \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{-nt}{n+1} \right)^k \sum_{j=1}^{k} \binom{k}{j} K_n^j, \quad t \geq 0,
\]

\[
(2.5) \quad K_n = \frac{2n}{n+1} \left\{ I - \frac{n-1}{n+1} V \right\}^{-1} V, \quad \text{and so } K_n(\mathcal{X}) \subseteq R, \quad n \geq 1.
\]

Reciprocally, we find from (2.3)(b) the following expression for \( V^n \) in terms of the \( S_t \):

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\[\text{\footnote{The absolute value sign refers to the usual Banach norm of the operator.}}\]

\[\text{\footnote{We will have } R = \mathcal{X} \text{ if and only if } H \text{ is self-adjoint, which in turn will be the case if and only if the isometries } S_t \text{ are actually unitary. For us this is the uninteresting case in which the Wold decomposition reduces to the triviality } \mathcal{X} = \mathcal{X}.}\]
\[ V^n = I + 2 \int_0^\infty L_n'(2t)e^{-t}Stdt, \]
(2.6)
\[ L_n(t) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} t^k, \quad \text{(nth Laguerre polynomial, [8]).} \]

From (2.5), (2.6) we get the following useful identity between the subspaces generated by the sets \( S_t(X), t \geq 0, \) and \( V^k(X), k \geq 0, \)
\[ \mathcal{E}\{S_t(X)\}_{t \geq 0} = \mathcal{E}\{V^k(X)\}_{k \geq 0}, \quad X \subseteq \mathfrak{A}. \]
From (2.5) we also see that
\[ S_t(x) = e^{-t}x + y_t, \quad y_t \in R, t \geq 0, \quad x \in \mathfrak{A}, \]
and hence
\[ (S_t(x), y) = e^{-t}(x, y), \quad x \in \mathfrak{A}, y \in R^1, t \geq 0, \]
\[ (S_t(x), S_t(y)) = e^{-1-t-l}(x, y), \quad x, y \in R^1, s, t \geq 0, \]
where \((, , )\) denotes the inner product in \( \mathfrak{A}. \)

3. The remote subspace. Let us write
\[ \mathfrak{A}_0 = \mathfrak{A}, \quad \mathfrak{A}' = \mathfrak{A}', \]
\[ \mathfrak{A}_\infty = \bigcap_{t \geq 0} \mathfrak{A}_t, \quad \mathfrak{A}'_\infty = \bigcap_{k \geq 0} \mathfrak{A}'_k. \]

We assert the following crucial theorem:

3.2. Theorem. \( \mathfrak{A}_\infty = \mathfrak{A}'_\infty. \) The restrictions of the isometries \( S_t, V^n, \) for \( t, n \geq 0, \) to the subspace \( \mathfrak{A}_\infty \) are unitary.

To prove this we first show, quite easily, that the restrictions of \( S_t \) and \( V^k \) to the remote subspaces \( \mathfrak{A}_\infty, \mathfrak{A}'_\infty, \) respectively, are unitary. We then establish the deeper result \( \mathfrak{A}_\infty = \mathfrak{A}'_\infty. \) The inclusion \( \mathfrak{A}_\infty \subseteq \mathfrak{A}'_\infty \) follows without much difficulty from (2.8) and (2.3). The reverse inclusion requires the following lemma, which rests on the fact that \( \mathfrak{D} \) is the range of \( I - V, \) and on the limiting behavior of \( L_n(t), \) as \( n \to \infty, \) cf. [8, pp. 333–334]:

Lemma. Let \( \mathfrak{D}_\infty = \bigcap_{k \geq 0} V^k(\mathfrak{D}), \) where \( \mathfrak{D} \) is the domain of the infinitesimal generator \( iH. \) Then
(a) \( \mathfrak{D}_\infty \) is a linear manifold everywhere dense in \( \mathfrak{A}'_\infty, \)
(b) \( \mathfrak{D}_\infty \subseteq \mathfrak{A}_\infty \) (and so \( \mathfrak{A}_\infty = \text{clos}. \mathfrak{D}_\infty \subseteq \mathfrak{A}_\infty). \)

Let us take the Wold decomposition (1.1) of \( \mathfrak{A} \) due to \( V = c(H), \)
the Cayley transform of \( H. \) As just shown \( \bigcap_{k \geq 0} V^k(\mathfrak{A}) = \mathfrak{A}_\infty. \) Also, on taking \( X = R^1 \) in (2.7) we find that \( \sum_{k=0}^\infty V^k(R^1) = \mathcal{E}\{V^k(R^1)\}_{k \geq 0} = \mathcal{E}\{S_t(R^1)\}_{t \geq 0}. \) Thus (1.1) reduces to
\[ \mathfrak{X} = \mathcal{E}\{S_t(R^\perp)\}_{t \in [0, \infty]} + \mathfrak{X}_\infty, \quad \mathcal{E}\{S_t(R^\perp)\}_{t \in [0, \infty]} \perp \mathfrak{X}_\infty. \]

On applying \( S_a \) we get
\[ (3.3) \quad \mathfrak{X}_a = \mathcal{E}\{S_t(R^\perp)\}_{t \in [0, \infty]} + \mathfrak{X}_\infty, \quad (a \geq 0), \quad \mathcal{E}\{S_t(R^\perp)\}_{t \in [0, \infty]} \perp \mathfrak{X}_\infty. \]

We shall refer to (3.3) as the pre-Wold decomposition of \( \mathfrak{X}_a \) due to the semi-group \( (S_t, t \geq 0) \). Our task is to express the first subspace on the right-hand side as a direct integral of differential subspaces.

4. Differential innovation subspaces. We first introduce an operator-valued interval-measure. The measure \( T_{ab} \) of the interval \([a, b]\), \( 0 \leq a \leq b \), is defined by
\[ (4.1) \quad T_{ab} = T_b - T_a, \quad \text{where} \quad T_t = \frac{1}{\sqrt{2}} \left\{ S_t - I - \int_0^t S_s ds \right\}, \quad t \geq 0. \]

We see at once that \( T_{ab}, T_t \) are bounded linear operators on \( \mathfrak{X} \) into \( \mathfrak{X} \), that \( T_0 = T_{0t}, t \geq 0 \), and that
\[ (a) \quad T_{ab} + T_{bc} = T_{ac}, \quad 0 \leq a \leq b \leq c, \]
\[ (b) \quad T_{ab} = \frac{1}{\sqrt{2}} \left\{ S_b - S_a - \int_a^b S_s ds \right\}, \quad 0 \leq a \leq b \]
\[ (c) \quad S_t T_{ab} = T_{a+t,b+t}, \quad 0 \leq a \leq b, \quad 0 \leq t. \]

By inverting the relations (4.1) we get the following expression for \( S_t \) in terms of the \( T_{or} \):
\[ (4.3) \quad S_t = -\sqrt{2} \int_t^\infty e^{-s}T_s ds = \sqrt{2} \left\{ T_t - \int_t^\infty e^{-s}T_s ds \right\}. \]

We consider next the subspace-valued interval measure:
\[ (4.4) \quad \mathfrak{M}_{ab} = T_{ab}(R^\perp), \quad 0 \leq a \leq b. \]

This has the following convenient properties, which are easy to check:
\[ (a) \quad S_t(\mathfrak{M}_{ab}) = \mathfrak{M}_{a+t,b+t}, \quad 0 \leq a \leq b, \quad 0 \leq t; \]
\[ (b) \quad \mathfrak{M}_{ab} \perp \mathfrak{M}_{cd}, \quad 0 \leq a < b \leq c < d; \]
\[ (c) \quad \frac{1}{\sqrt{b-a}} T_{ab} \text{ is an isometry on } R^\perp \text{ onto } \mathfrak{M}_{ab}, \quad a < b; \]
\[ \text{i.e. } (T_{ab}x, T_{ab}y) = (b-a)(x, y), \quad x, y \in R^\perp; \]
\[ (d) \quad (T_J(x), T_K(y)) = (T_J \cap T_K(x), T_J \cap T_K(y)) = \left| J \cap K \right| (x, y), \]
where \( x, y \in R^\perp, \ J, \ K \text{ are intervals and } \left| \right| \text{ is the length.} \]
From (4.5)(c) we see at once that

\( \mathcal{N}_{ab} \) is a (closed) subspace of \( \mathcal{X} \), and \( \dim \mathcal{N}_{ab} = \dim R^1 \), \( 0 \leq a < b \).

But it should be noted that our subspace-valued measure \( \mathcal{N}_{ab} \) is only subadditive, i.e. \( \mathcal{N}_{ac} \subset \mathcal{N}_{ab} + \mathcal{N}_{bc}, 0 \leq a < b < c \); for, we find that

\[
(4.7) \quad \mathcal{N}_{ae} \cap (\mathcal{N}_{ab} + \mathcal{N}_{bc}) = \left( \frac{1}{b-a} T_{ab} - \frac{1}{c-b} T_{bc} \right) (R^1),
\]

and the last is not \( \{0\} \) even when \( \dim R^1 = 1 \).

A simple but important consequence of (4.2)(b) and (4.3) is the identity

\[
(4.8) \quad \mathcal{N} \{ T_t(R^1) \}_{t \geq a} = \mathcal{N} \{ \mathcal{N}_{st} \}_{s \geq t < \infty} = \mathcal{N} \{ T_{st}(R^1) \}_{s \geq t < \infty}.
\]

This identity enables us to restate the pre-Wold decomposition (3.3) in the form

\[
(4.9) \quad \mathcal{X}_a = \mathcal{N} \{ T_{st}(R^1) \}_{s \geq t < \infty} + \mathcal{X}_\infty, \quad (a \geq 0), \quad T_{st}(R^1) \perp \mathcal{X}_\infty.
\]

On comparing this with the corresponding decomposition in the discrete case (cf. (1.1), (3.1)), viz.

\[
\mathcal{X}_n = \mathcal{N} \{ V^k(R^1) \}_{k \geq n} + \mathcal{X}_\infty', \quad (n \geq 0), \quad V^k(R^1) \perp \mathcal{X}_\infty',
\]

we see that the subspaces \( T_{st}(R^1) \) have taken the place of the "innovation subspaces" \( V^k(R^1) \). This fact along with the properties (4.5)(b), (4.6) justifies our calling \( T_{st}(R^1), 0 \leq s < t \), the differential innovation subspaces of \( \mathcal{X} \) engendered by the semi-group \( (S_t, t \geq 0) \).

Now in the discrete case we have the direct sum representation:

\[
\mathcal{N} \{ V^k(R^1) \}_{k \geq 0} = \sum_{k=0}^{\infty} V^k(R^1),
\]

where, by definition,

\[
\sum_{k=0}^{\infty} V^k(R^1) = \left\{ \xi: \xi = \sum_{k=0}^{\infty} V^k(x_k), x_k \in R^1 \& \sum_{k=0}^{\infty} |x_k|^2 < \infty \right\}.
\]

This suggests that in the continuous case we should have an analogous direct integral representation:

\[
\mathcal{N} \{ (T_{st}(R^1))_{0 \leq s < t < \infty} = \int_0^{\infty} T_{dt}(R^1),
\]

where

\[
\int_0^{\infty} T_{dt}(R^1) = \left\{ \xi: \xi = \int_0^{\infty} T_{dt}(x_t), x_t \in R^1 \& \int_0^{\infty} |x_t|^2 dt < \infty \right\}.
\]
This heuristic reasoning can be put on a sound footing by defining precisely the vector-valued integral $\int_0^a T_{dt}(x_t)$ occurring in the last equation. This is done in §§5, 6 below.

5. Generalized vector-valued integrals. Let $L_2([a, b], \mathbb{R}^1)$ be the Hilbert space of all strongly (Lebesgue) measurable functions $x$ on $[a, b]$ with values $x_t \in \mathbb{R}^1$ such that $\int_a^b |x_t|^2 dt < \infty$. Our task is to define $\int_0^a T_{dt}(x_t)$ so that it will behave like a vector sum $\sum_{k=1}^n V_k(x_k)$, where $x_k \in \mathbb{R}^1$. This suggests that we define it so as to ensure the following properties: for all functions $x, y, x^{(n)} \in L_2([a, b], \mathbb{R}^1)$,

(a) $\left( \int_a^b T_{dt}(x_t), \int_a^b T_{dt}(y_t) \right) = \int_a^b (x_t, y_t) dt$,

(b) $\left| \int_a^b T_{dt}(x_t) \right|^2 = \int_a^b |x_t|^2 dt$,

(c) $\int_a^b T_{dt}(cx_t + dy_t) = c \int_a^b T_{dt}(x_t) + d \int_a^b T_{dt}(y_t)$,

(d) $\int_a^b T_{dt}(x^{(n)}_t) \to \int_a^b T_{dt}(x_t)$, when $x^{(n)} \to x$ in the $L_2$-topology.

The requisite definition consists of two parts, one for step-functions $x$ and the other for arbitrary $x$ in $L_2([a, b], \mathbb{R}^1)$:

5.2(a). Definition. For the step-function $x = \sum_{k=1}^n \alpha_k \chi_{J_k}$ on $[a, b]$, where $\alpha_k \in \mathbb{R}^1$ and $\chi_{J_k}$ is the indicator-function of the bounded interval $J_k$, we define $f_0^a T_{dt}(x_t) = \sum_{k=1}^n T_{dt}(\alpha_k)$.

It follows from (4.5) that this definition is unequivocal and that the laws (5.1)(a)-(c) hold when $x$ and $y$ are step-functions. Moreover, for any Cauchy-sequence of step-functions $x^{(n)}$ in $L_2([a, b], \mathbb{R}^1)$ we have

$$\left| \int_a^b T_{dt}(x^{(m)}_t) - \int_a^b T_{dt}(x^{(n)}_t) \right|^2 = \int_a^b |x^{(m)}_t - x^{(n)}_t|^2 dt \to 0,$$

as $m, n \to \infty$. This relation and the well-known fact that the step-functions are everywhere dense in $L_2([a, b], \mathbb{R}^1)$ suggest the following extension of our definition:

5.2(b). Definition. For any $x \in L_2([a, b], \mathbb{R}^1)$, we define $f_0^a T_{dt}(x_t) = \lim_{n \to \infty} \int_a^b T_{dt}(x^{(n)}_t)$, where $(x^{(n)}_t, n \geq 1)$ is any sequence of step-functions tending to $x$ in the $L_2$-topology.

It is easy to check that our definition is again unequivocal, and

---

* Cf. [3, Chapter III, §6]. According to their Theorem 6, $L_2([a, b], \mathbb{R}^1)$ is a Banach space. With the inner product $(x, y) = \int_a^b (x_t, y_t) dt$, it is obviously a Hilbert space.
that the laws (5.1) hold without restriction. Moreover, as an interval-
function the integral is seen to have the following properties:

\[(a) \int_a^b T_{dt}(x_t) + \int_b^c T_{dt}(x_t) = \int_a^c T_{dt}(x_t), \quad 0 \leq a < b < c,\]

\[(b) \int_J T_{dt}(x_t) \perp \int_K T_{dt}(y_t), \quad J, K \text{ non overlapping,}\]

\[(5.3) \quad (\int_J T_{dt}(x_t), \int_K T_{dt}(y_t)) = \int_{J \cap K} (x_t, y_t) dt,\]

\[(c) S_{c}\left\{\int_a^b T_{dt}(x_t)\right\} = \int_{a+c}^{b+c} T_{ds}(x_{t-c})\).

From (5.1) and (5.3) we see that our vector-valued integral has
properties akin to those possessed by stochastic integrals.\(^7\) To see the
precise relationship between the two concepts, consider the function
\[x_t = c(t)\alpha, \quad \alpha \in \mathbb{R}^1,\] and let \[\xi_t = T_i(\alpha).\] Then it follows easily that the process
\[(\xi_t, t \geq 0)\] has orthogonal increments, and

\[(5.5) \quad \int_a^b T_{dt}\{c(t)\alpha\} = \int_a^b c(t)d\xi_t \quad \text{(stochastic integral).}\]

This shows that our notion of vector-integration subsumes that of
stochastic integration, but reduces to the latter when and only when
\[\text{dim. } \mathbb{R}^L = 1.\]

6. The direct integral. We can now define our direct integral as a
set of vector-valued integrals:

\[(6.1) \quad \int_a^b T_{dt}(\mathbb{R}^L) = \left\{\xi: \xi = \int_a^b T_{dt}(x_t), x \in L_\mathbb{R}([a, b], \mathbb{R}^L)\right\},\]

where \[0 \leq a < b.\] By (5.1)(c), (d) this integral is a (closed) subspace
of \(X.\) Indeed, (5.1) enables us at once to assert the following theorem:

6.2. Theorem. The correspondence \(x \rightarrow \int_a^b T_{dt}(x_t)\) is an isomorphism
on the Hilbert space \(L_\mathbb{R}([a, b], \mathbb{R}^L)\) onto the subspace \(\int_a^b T_{dt}(\mathbb{R}^L)\) of \(X,\)
\[0 \leq a < b.\]

From (5.3) we see, moreover, that as an interval-function our

\(^7\) Such integrals were introduced in probability theory by Wiener, Cramer and
Doob. They also occur in Hilbert space theory when spectral integrals \(\int_\lambda^\infty c(\lambda)dE_\lambda,\)
where \((E_\lambda, a \leq \lambda \leq b)\) is a resolution of \(I,\) are applied to vectors. Cf. [2, Chapter IX, §2], and [9, Chapter VI, §2].
direct integral has the following convenient properties:

(a) \( \int_{a}^{b} T_{dt}(R^1) + \int_{b}^{c} T_{dt}(R^1) = \int_{a}^{c} T_{dt}(R^1), \ 0 \leq a < b < c, \)

(b) \( \int_{J} T_{dt}(R^1) \perp \int_{K} T_{dt}(R^1), \ J, K \) nonoverlapping,

\[ (6.3) \]

(c) \( \int_{J} T_{dt}(R^1) \subseteq \int_{K} T_{dt}(R^1), \ J \subseteq K, \)

(d) \( S_{\varepsilon} \left\{ \int_{a}^{b} T_{dt}(R^1) \right\} = \int_{a+\varepsilon}^{b+\varepsilon} T_{dt}(R^1). \)

We can also show that

\[ (6.4) \]

This relation with \( b = \infty \) together with (4.9) yields the result we had set out to prove:

6.5. Theorem (Wold decomposition). Let \((S_{t}, t \geq 0)\) be a strongly continuous semi-group of isometries on \( \mathcal{H} \) into \( \mathcal{H} \), \( iH \) be its infinitesimal generator and \( V \) the Cayley transform of \( H \). Then for \( a \geq 0 \)

\[ S_{a}(\mathcal{H}) = \int_{0}^{a} T_{dt}(R^1) + \mathcal{H}_{\infty}, \quad \int_{0}^{\infty} T_{dt}(R^1) \perp \mathcal{H}_{\infty}, \]

where \( R = V(\mathcal{H}) \) and \( \mathcal{H}_{\infty} = \bigcap_{t \geq 0} S_{t}(\mathcal{H}). \)

From this decomposition we can readily obtain Cooper's theorem that our semi-group can be embedded in a unitary group acting on a larger Hilbert space [1, p. 841].

Our direct integral does not bear any obvious relation to the direct integral \( \int_{a}^{\infty} \mathcal{H} d\mu(t) \) due to von Neumann and others, cf. [7], in which \( \mathcal{H} \) is a Hilbert space and \( \mu \) a complex-valued measure. Our integral could be written in the form \( \int_{a}^{\infty} \mathcal{H} dt \) on letting \( \mathcal{H}_{t} = T_{0t}(R^1), \) cf. (4.4). But the significant factor in its definition is the family of operators \( T_{0t} \) and not the family of subspaces \( \mathcal{H}_{t}, \) cf. Definitions 5.2(a), (6.1). It would seem that this integral is the tool needed for the study of the isometric representations of continuous semi-groups, just as the von Neumann integral is the tool required to deal with the unitary representations of continuous groups.

References


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