RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ANALYTIC MEASURES ON COMPACT GROUPS

BY K. DE LEEUW AND I. GLICKSBERG

Communicated by Ralph Phillips, August 14, 1962

The purpose of this note is the announcement of an extension to compact abelian groups of the two celebrated theorems of F. and M. Riesz [8] concerning analytic measures on the circle group. The content of these theorems is as follows:

Let \( \mu \) be a Borel measure on the circle satisfying

\[
\int_{-\pi}^{\pi} e^{in\theta} d\mu(\theta) = 0, \quad n = 1, 2, 3, \ldots
\]

Then

A. \( \mu \) is absolutely continuous with respect to Lebesgue measure and

B. If \( \mu \) vanishes identically on a set of positive Lebesgue measure, then \( \mu \) must be the zero measure.

It is not hard to see that A and B together are equivalent to the following:

The collection of Borel sets on which \( \mu \) vanishes identically is invariant under rotation.

This is the assertion concerning analytic measures that we extend to compact groups. We also shall state several of its consequences, including analogues of A and B. The work was inspired by, and is in part an extension of, several of the results of Helson and Lowdenslager [4; 5].

In all that follows \( G \) is a compact abelian group, \( \hat{G} \) its discrete dual, and \( \psi \) is a fixed homomorphism of \( \hat{G} \) into the group \( \mathbb{R} \) of real numbers. The mapping \( \psi: \hat{G} \rightarrow \mathbb{R} \) is a continuous homomorphism and

\footnote{Supported in part by National Science Foundation Grant G14779 and the United States Air Force Office of Scientific Research.}

\footnote{We say that a measure \( \mu \) vanishes identically on a set \( E \) if \( \mu \) vanishes on all Borel subsets of \( E \).}

\footnote{See however our final remarks.}
ANALYTIC MEASURES ON COMPACT GROUPS

thus induces a continuous homomorphism $\phi: R \to G$ of the associated
dual groups; $\phi$ is the unique mapping of $R$ into $G$ satisfying

$$\sigma(\phi(t)) = e^{i\phi(t)}e^t, \quad t \in R, \quad \sigma \in \hat{G}.$$ 

We shall use $\hat{\ }$ to denote Fourier transform, $\ast$ to denote convolu-
tion and by measure we shall mean finite complex regular Borel meas-
ure. If $\mu$ is a measure, $|\mu|$ is the associated total variation measure.

A function or measure on $G$ is called $\phi$-analytic if its Fourier trans-
form vanishes on $\{\sigma: \sigma \in \hat{G}, \psi(\sigma) < 0\}$. A measure $\mu$ on $G$ is called
quasi-invariant under $\phi$ if $\{E: E$ Borel, $|\mu|(E) = 0\}$ is invariant under
translation by the elements of $\phi(R)$.

**Main Theorem.** Let $\mu$ be a $\phi$-analytic measure. Then $\mu$ is quasi-
invariant under $\phi$.

Denote by $\rho$ the image of the measure $(1/1+x^2)dx$ on $R$ under the
mapping $\phi: R \to G$. It is not hard to show that a measure $\mu$ on $G$ is quasi-
invariant under $\phi$ if and only if $|\mu|$ and $\rho \ast |\mu|$ are mutually ab-
solutely continuous. Thus we have a reformulation.

**Main Theorem.** Let $\mu$ be a $\phi$-analytic measure on $G$. Then $|\mu|$ and
$\rho \ast |\mu|$ are mutually absolutely continuous.

Before stating the first consequences of this result some further
definitions are necessary. If $E$ is a Borel subset of $G$ we shall say that
$E$ is of measure zero in the direction of $\phi$ of each coset $x + \phi(R)$ inter-
sects $E$ in a set of linear measure zero; more precisely, if for each $x$ in $G$,

$$\{t: t \in R, x + \phi(t) \in E\}$$

has Lebesgue measure zero. A measure $\mu$ on $G$ that vanishes on each
subset of $G$ which is of measure 0 in the direction of $\phi$ is called abso-
lutely continuous in the direction of $\phi$. It can be shown that $\mu$ is ab-
solutely continuous in the direction of $\phi$ if and only if it translates
continuously in the direction of $\phi$; that is, if

$$\lim_{t \to 0} ||\mu_t - \mu|| = 0,$$

where $||\cdot||$ is the total variation norm, and for each $t$ in $R$ the trans-
lated measure $\mu_t$ is defined by

$$\mu_t(E) = \mu(\phi(t) + E), \quad E$ Borel.$$

(For the circle group this result is due to Plessner [7].)

A measure quasi-invariant under $\phi$ is easily shown to be absolutely
continuous in the direction of \( \phi \), so by the Main Theorem we have
the following analogue of assertion A above.

**Theorem A.** Let \( \mu \) be a \( \phi \)-analytic measure on \( G \). Then \( \mu \) is absolutely
continuous in the direction of \( \phi \).

For \( \mu \) a measure on \( G \), the \( \psi \)-conjugate of \( \mu \) is defined to be that
measure \( \mu_\psi \) (if such exists) whose Fourier transform satisfies

\[
\mathcal{O}_\psi(\sigma) = \begin{cases} 
\mathcal{O}(\sigma), & \psi(\sigma) > 0 \\
0, & \psi(\sigma) = 0 \\
-\mathcal{O}(\sigma), & \psi(\sigma) < 0.
\end{cases}
\]

Theorem A is equivalent to the assertion that each measure on \( G \)
having a \( \psi \)-conjugate is absolutely continuous in the direction of \( \phi \).

Theorem A together with the result of Bishop [1] yields the follow­
ing, which for the circle group is due to Rudin [9] and Carleson [3].

**Corollary 1.** Let \( E \) be a closed subset of \( G \). Then the following are
equivalent:
1°. \( E \) is of measure zero in the direction of \( \phi \).
2°. For each continuous function \( g \) on \( E \) there is a continuous \( \phi \)-
analytic function \( f \) on \( G \) that agrees on \( E \) with \( g \).

If \( H \) is the \( n \)-torus, its dual \( \hat{H} \) is the group of lattice points in real
\( n \)-space. Bochner's extension of the F. and M. Riesz Theorem (see
[2]) states that any measure on the \( n \)-torus whose Fourier transform
vanishes off the positive octant of the lattice points must be ab­solutely continuous. Theorem A applied \( n \)-times yields the following,
which includes the Bochner Theorem.

**Corollary 2.** Let \( H \) be the \( n \)-torus, \( \mu \) a measure on \( H \) and \( F \) a set of
\( n \) homomorphisms of \( \hat{H} \) into \( R \). Assume that the set \( F \) is linearly inde­
pendent and that for each \( \psi \) in \( F \) the conjugate measure \( \mu_\psi \) exists. Then
\( \mu \) must be absolutely continuous.

One further definition is necessary before we can state our extension
of assertion B above. For \( E \) a Baire subset of \( G \) we denote by \( E_\phi \) the
union of all cosets \( x + \phi(R) \) that intersect \( E \) in a set of positive linear
measure. More precisely, \( E_\phi \) consists of those \( x \) in \( G \) for which

\[
\{t: t \in R, \ x + \phi(t) \subseteq E\}
\]

has positive Lebesgue measure.

**Theorem B.** Let \( \mu \) be a \( \phi \)-analytic measure on \( G \). Suppose that \( E \) is a
Baire subset of \( G \) on which \( \mu \) vanishes identically. Then \( \mu \) vanishes identi­
tically on \( E_\phi \).
As a special case we have the following result, the second half of which is due to Helson, Lowdenslager and Malliavin [5].

**Corollary 3.** Assume that \( \phi(R) \) is dense in \( G \). Let \( \mu \) be a \( \phi \)-analytic measure on \( G \) that either

1. vanishes identically on an open subset of \( G \)
2. is absolutely continuous with respect to Haar measure and vanishes identically on a Borel set of positive Haar measure.

Then \( \mu \) is the zero measure.

The next result is a simple consequence of the Main Theorem. A special case of the proposition has also been obtained by Frank Forelli using quite different methods. The corollaries that we list are in part refinements of results of Helson-Lowdenslager [4] and Bochner [2]. Bochner has informed us that he has been able to obtain the corollaries using the results of [4].

**Proposition.** Let \( \mu \) be a \( \phi \)-analytic measure on \( G \) (or more generally, any measure on \( G \) quasi-invariant under \( \phi \)). Let \( \eta \) be a measure on \( G \) that is the image of some measure on \( R \) under the mapping \( \phi: R \to G \). Then the convolution \( \eta \ast \mu \) is absolutely continuous with respect to \( \mu \). In particular, if \( \mu \) is singular with respect to Haar measure, \( \eta \ast \mu \) is either singular with respect to Haar measure or is the zero measure.

To simplify the statements of the corollaries we assume that \( \hat{G} \) is \( R \) with the discrete topology and \( \psi: \hat{G} \to R \) the identity mapping, so that \( G \) is the Bohr compactification of the reals.

**Corollary 4.** Let \( K \) be a closed subset of \( R \). Let \( \mu \) be a \( \phi \)-analytic measure on \( G \), \( \lambda \) its singular part. If \( \mu \) vanishes off \( K \) then \( \lambda \) also vanishes off \( K \).

For \( K = \{ t: t \leq 0 \} \), this is due to Helson-Lowdenslager [4].

**Corollary 5.** Let \( \mu \) be a singular \( \phi \)-analytic measure on \( G \). Then \( \{ \sigma: \hat{\mu}(\sigma) \neq 0 \} \) is a subset of \( R \) containing no isolated points.

**Corollary 6.** Let \( K \) be a countable closed subset of \( R \). Let \( \mu \) be a \( \phi \)-analytic measure on \( G \) whose Fourier transform vanishes off \( K \). Then \( \mu \) is absolutely continuous with respect to Haar measure.

There are several questions connected with the above results which deserve mention. First, most of our deductions from the Main Theorem are valid in the context of one-parameter groups of homeomorphisms of compact topological spaces. It is conceivable that a version of the Main Theorem itself is also valid in this context. Here is a
possible generalization. Let $X$ be a compact space and $\{T_t\}$ a one-parameter group of homeomorphisms of $X$. Call a measure $\mu$ on $X$ $\{T_t\}$-analytic if the vector valued integral

$$\int_{-\infty}^{+\infty} h(t)T_t \mu dt$$

is zero for all $h$ in $L^1(\mathbb{R})$ whose Fourier transforms vanish for $t \leq 0$. (In the case that $X = G$ and $T_t$ is translation by $\phi(t)$, this agrees with our previous definition of analyticity.) Then a generalization of our Main Theorem would be the assertion that a $\{T_t\}$-analytic measure $\mu$ is quasi-invariant under $\{T_t\}$; that is, the collection of $|\mu|$-null sets is $\{T_t\}$ invariant. Indeed, with this definition of analyticity (and $T_t$ translation by $\phi(t)$), the Main Theorem continues to hold even when $G$ is noncommutative.

Another possible extension of some of our results is to the context of Dirichlet algebras (for the relevant definitions, see [6]). The collection of $\phi$-analytic continuous functions on $G$ is a Dirichlet algebra on $G$. Theorem A says precisely that each Borel subset of $G$ that is of measure zero for all of the representing measures for the algebra must be of measure zero for all of the annihilating measures for the algebra. It is conceivable that a corresponding result holds for a wider class of Dirichlet algebras.

BIBLIOGRAPHY

2. S. Bochner, Boundary values of analytic functions in several variables and of almost periodic functions, Ann. of Math. 45 (1944), 708–722.